

# Regular representations and Huang-Lepowsky tensor functors for vertex operator algebras

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## Abstract

This is the second paper in a series to study regular representations for vertex operator algebras. In this paper, given a module  $W$  for a vertex operator algebra  $V$ , we construct, out of the dual space  $W^*$ , a family of canonical (weak)  $V \otimes V$ -modules called  $\mathcal{D}_{Q(z)}(W)$  parametrized by a nonzero complex number  $z$ . We prove that for  $V$ -modules  $W, W_1$  and  $W_2$ , a  $Q(z)$ -intertwining map of type  $\binom{W'}{W_1 W_2}$  in the sense of Huang and Lepowsky exactly amounts to a  $V \otimes V$ -homomorphism from  $W_1 \otimes W_2$  to  $\mathcal{D}_{Q(z)}(W)$  and that a  $Q(z)$ -tensor product of  $V$ -modules  $W_1$  and  $W_2$  in the sense of Huang and Lepowsky amounts to a universal from  $W_1 \otimes W_2$  to the functor  $\mathcal{F}_{Q(z)}$ , where  $\mathcal{F}_{Q(z)}$  is a functor from the category of  $V$ -modules to the category of weak  $V \otimes V$ -modules defined by  $\mathcal{F}_{Q(z)}(W) = \mathcal{D}_{Q(z)}(W')$  for a  $V$ -module  $W$ . Furthermore, Huang-Lepowsky's  $P(z)$  and  $Q(z)$ -tensor functors for the category of  $V$ -modules are extended to functors  $T_{P(z)}$  and  $T_{Q(z)}$  from the category of  $V \otimes V$ -modules to the category of  $V$ -modules. It is proved that functors  $\mathcal{F}_{P(z)}$  and  $\mathcal{F}_{Q(z)}$  are right adjoints of  $T_{P(z)}$  and  $T_{Q(z)}$ , respectively.

## 1 Introduction

Let  $V$  be a vertex operator algebra. In [Li2], for a  $V$ -module  $W$  and a nonzero complex number  $z$  we constructed a weak  $V \otimes V$ -module called  $\mathcal{D}_{P(z)}(W)$  out of the dual space  $W^*$ . We proved that a  $P(z)$ -intertwining map of type  $\binom{W'}{W_1 W_2}$  in the sense of [H3, HL1-4], as proved in [HL2], which is the evaluation of an intertwining operator, exactly amounts to a  $V \otimes V$ -homomorphism from  $W_1 \otimes W_2$  into  $\mathcal{D}_{P(z)}(W)$  ( $\subset W^* = \overline{W'}$ ). We furthermore proved that for  $V$ -modules  $W_1$  and  $W_2$ , a  $P(z)$ -tensor product in the sense of [H3, HL1-4] exactly amounts to a universal from  $W_1 \otimes W_2$  to the functor  $\mathcal{F}_{P(z)}$ , where  $\mathcal{F}_{P(z)}$  is a functor from the category of  $V$ -modules to the category of  $V \otimes V$ -modules defined by  $\mathcal{F}_{P(z)}(W) = \mathcal{D}_{P(z)}(W')$  for a  $V$ -module  $W$ .

The objects  $P(z)$  and  $Q(z)$  are two special elements of the moduli space  $K$  of spheres with (ordered) punctures and local coordinates vanishing at those punctures (see [HL2]), where the three ordered punctures are  $\infty, z, 0$  for  $P(z)$  and  $z, \infty, 0$  for  $Q(z)$  with the standard local coordinates. In the tensor product theory developed in [H3, HL1-4] for a vertex operator algebra  $V$ ,  $P(z)$  and  $Q(z)$ -tensor functors were constructed. As remarked in [HL2], these different tensor functors will play important roles in the formulation and constructions of the associativity and commutativity isomorphisms.

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In this paper, a sequel to [Li2], we shall obtain parallel results (with  $P(z)$  being replaced by  $Q(z)$ ); we construct a weak  $V \otimes V$ -module called  $\mathcal{D}_{Q(z)}(W)$  (out of  $W^*$ ) such that a  $Q(z)$ -intertwining map of type  $\binom{W'}{W_1 W_2}$  in the sense of Huang-Lepowsky exactly amounts to a  $V \otimes V$ -homomorphism from  $W_1 \otimes W_2$  into  $\mathcal{D}_{Q(z)}(W)$  and such that a  $Q(z)$ -tensor product of  $V$ -modules  $W_1$  and  $W_2$  in the sense of Huang-Lepowsky exactly amounts to a universal from  $W_1 \otimes W_2$  to a functor  $\mathcal{F}_{Q(z)}$ .

Just like in the tensor product theory ([H3],[HL1-5]), each of the  $P(z)$  and  $Q(z)$ -notions has its own advantages in practice. For example, the  $Q(z)$ -notions are more natural in considering trace functions. In fact, in the study of genus-zero correlation functions [Li3], it is more natural to use the notion of  $\mathcal{D}_{Q(z)}(W)$ .

As implied by the results of [Li2], the functor  $\mathcal{F}_{P(z)}$  is essentially a right adjoint of Huang-Lepowsky  $P(z)$ -tensor functor. To make this statement precise, one needs to extend Huang-Lepowsky  $P(z)$ -tensor functor to a functor from the category of  $V \otimes V$ -modules to the category of  $V$ -modules. (Note that the product category of the category of  $V$ -modules is a natural subcategory of the category of  $V \otimes V$ -modules.) In this paper, we carry out this exercise to extend Huang-Lepowsky's  $P(z)$  and  $Q(z)$ -tensor functors for the category of  $V$ -modules to functors  $T_{P(z)}$  and  $T_{Q(z)}$ , respectively, from the category  $\mathcal{C}_{V \otimes V}^o$  to the category of  $V$ -modules, where  $\mathcal{C}_{V \otimes V}^o$  consists of  $V \otimes V$ -modules on which  $L(0) \otimes 1$  and  $1 \otimes L(0)$  act semisimply. It is proved that if  $V$  is rational in the sense of [HL1-4], functors  $\mathcal{F}_{P(z)}$  and  $\mathcal{F}_{Q(z)}$  are exactly right adjoints of  $T_{P(z)}$  and  $T_{Q(z)}$ , respectively. In establishing the functors  $T_{P(z)}$  and  $T_{Q(z)}$  (Theorems 5.7 and 6.8), instead of following [HL1-4] ([HL2], Theorem 6.1 and [HL4], Theorem 13.9), we use a different approach by employing the regular representations. On the other hand, presumably Theorems 5.7 and 6.8 follow from [HL1-4] straightforward without any further efforts. Furthermore, we have greatly taken the advantage of [HL1-4] and we are motivated by their results for Propositions 3.4 and 6.7 and Theorem 6.8.

As mentioned in [Li2], there is a classical analogue in the representation theory of Lie algebras for the left-right adjointness phenomenon. (We thank James Lepowsky for bringing up this classical analogue issue.) Let  $\mathfrak{g}$  be a Lie algebra and  $U(\mathfrak{g})$  the universal enveloping algebra. Usually,  $U(\mathfrak{g})$  as a left (right)  $U(\mathfrak{g})$ -module is called the left (right) regular representation of  $\mathfrak{g}$  (cf. [Di]). Let  $\Delta$  be the diagonal map from  $\mathfrak{g}$  to the product Lie algebra  $\mathfrak{g} \times \mathfrak{g}$ . Since  $\Delta$  is a Lie algebra homomorphism, the diagonal map  $\Delta$  gives rise to a natural functor  $T$  from the category of  $\mathfrak{g} \times \mathfrak{g}$ -modules to the category of  $\mathfrak{g}$ -modules. Furthermore, the standard tensor functor of the category of  $\mathfrak{g}$ -modules is the restriction of this functor  $T$ . On the other hand, for any  $\mathfrak{g}$ -module  $U$ , we have a coinduced module  $\text{Hom}_{U(\mathfrak{g})}(U(\mathfrak{g} \times \mathfrak{g}), U)$  (cf. [Di]). Then the notion of coinduced module gives rise to a functor  $E$  from the category of  $\mathfrak{g}$ -modules to the category of  $\mathfrak{g} \times \mathfrak{g}$ -modules. From [Di] (Proposition 5.5.3), the coinduced module functor  $E$  is a right adjoint of the tensor functor  $T$ . (The usual induced module functor is a left adjoint of the tensor functor  $T$ .)

In the classical Lie theory, the Harish-Chandra bimodule theory is quite important. In [FM], Frenkel and Malikov studied Harish-Chandra categories by using Kazhdan-Lusztig tensor functor for affine Lie algebras [KL1-4] and obtained certain very interesting results. (Kazhdan-Lusztig tensor functor and Huang-Lepowsky's tensor functors for the

vertex operator algebras associated to affine Lie algebras are closely related.) In vertex operator algebra theory, analogues of many classical fundamental notions and theories have been carried out. On the other hand, due to the complexity of the new theory, certain (straightforward) analogues do not make any sense. A very interesting question is how much the classical Harish-Chandra bimodule theory can be carried out in the theory of vertex operator algebras. We may address this issue in the future.

This paper is organized as follows: In Section 2, we recall the construction of  $\mathcal{D}_{P(z)}(W)$  and the main results from [Li2]. Sections 3 and 4 are devoted to the construction of  $\mathcal{D}_{Q(z)}(W)$  and its relation with Huang-Lepowsky's notion of  $Q(z)$ -tensor product. In Sections 5 and 6, Huang-Lepowsky's  $Q(z)$  and  $P(z)$ -tensor functors are extended to functors  $T_{Q(z)}$  and  $T_{P(z)}$  and the left-right adjointness of functors are established.

## 2 The weak $V \otimes V$ -module $\mathcal{D}_{P(z)}(W)$

In this section, we shall review the construction of the weak  $V \otimes V$ -module  $\mathcal{D}_{P(z)}(W)$  and the main results obtained in [Li2]. This section does not contain any new result, however we shall use the explicit construction in later sections.

As in [Li2], we use standard definitions and notations as given in [FLM, FHL]. A vertex operator algebra is denoted by  $(V, Y, \mathbf{1}, \omega)$ , where  $\mathbf{1}$  is the vacuum vector and  $\omega$  is the Virasoro element, or simply by  $V$ . We also use the notion of weak module as defined in [DLM]—A weak module satisfies all the axioms defining the notion of a module given in [FLM] and [FHL] except those involving grading. Then no grading is required for a weak module.

Since this paper deals with only the representation theory of vertex operator algebras, *throughout this paper we fix a vertex operator algebra  $V$ .*

We typically use letters  $x, y, x_1, x_2, \dots$  for mutually commuting formal variables and  $z, z_0, \dots$  for complex numbers. For a vector space  $U$ ,  $U[[x, x^{-1}]]$  is the vector space of all (possibly doubly infinite) formal series with coefficients in  $U$ ,  $U((x))$  is the space of formal Laurent series and  $U((x^{-1}))$  is the space of formal series truncated from above. We shall also use the following standard formal variable convention (cf. [FLM]):

$$(x_1 - x_2)^n = \sum_{i \geq 0} (-1)^i \binom{n}{i} x_1^{n-i} x_2^i, \quad (2.1)$$

$$(x - z)^n = \sum_{i \geq 0} (-z)^i \binom{n}{i} x^{n-i}, \quad (2.2)$$

$$(z - x)^n = \sum_{i \geq 0} (-1)^i z^{n-i} \binom{n}{i} x^i \quad (2.3)$$

for  $n \in \mathbb{Z}$ ,  $z \in \mathbb{C}^\times$ .

For vector spaces  $U_1, U_2$ , a linear map  $f \in \text{Hom}(U_1, U_2)$  extends canonically to a linear map from  $U_1[[x, x^{-1}]]$  to  $U_2[[x, x^{-1}]]$ . We shall use this canonical extension without any comments.

Let  $M$  be a vector space and let  $G$  be a linear map from  $V$  to  $(\text{End } M)[[x, x^{-1}]]$ . For  $v \in V$ , we set (cf. [FHL, HL1])

$$G^o(v, x) = G(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}), \quad (2.4)$$

which lies in  $(\text{End } M)[[x, x^{-1}]]$  because  $e^{xL(1)}(-x^{-2})^{L(0)}v \in V[x, x^{-1}]$ . Then ([FHL, Proposition 5.3.1])

$$(G^o)^o = G. \quad (2.5)$$

In particular, if  $(W, Y)$  is a weak  $V$ -module, for  $v \in V$ ,

$$Y^o(v, x) = Y(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}) \in (\text{End } W)[[x, x^{-1}]]. \quad (2.6)$$

As pointed out in [HL2], the proof Theorem 5.2.1 of [FHL] proves that  $Y^o$  satisfies an opposite Jacobi identity (cf. (2.9)). It is a viewpoint of [HL1] that the pair  $(W, Y^o)$  should be thought of as a right (weak)  $V$ -module.

A *right* weak  $V$ -module is a vector space  $W$  equipped with a linear map  $Y_W$  from  $V$  to  $(\text{End } W)[[x, x^{-1}]]$  such that for  $v \in V$ ,  $w \in W$ ,

$$Y_W(v, x)w \in W((x^{-1})), \quad (2.7)$$

$$Y_W(\mathbf{1}, x) = 1, \quad (2.8)$$

and such that the following *opposite Jacobi identity* holds for  $u, v \in V$ :

$$\begin{aligned} & x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_W(v, x_2) Y_W(u, x_1) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) Y_W(u, x_1) Y_W(v, x_2) \\ &= x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y_W(Y(u, x_0)v, x_2). \end{aligned} \quad (2.9)$$

The following result formulated in [Li2], due to [FHL, HL2], gives the equivalence between the notion of (left)  $V$ -module and the notion of right  $V$ -module:

**Proposition 2.1** *Let  $W$  be a vector space equipped with a linear map  $Y_W$  from  $V$  to  $(\text{End } W)[[x, x^{-1}]]$ . Then  $(W, Y_W)$  is a left (weak)  $V$ -module if and only if  $(W, Y_W^o)$  is a right (weak)  $V$ -module.*

Let  $W$  be a weak  $V$ -module for now. Then

$$Y^o(v, x) \in \text{Hom}(W, W((x^{-1}))) \quad \text{for } v \in V. \quad (2.10)$$

Furthermore, for any complex number  $z_0$ ,

$$Y^o(v, x + z_0) \in \text{Hom}(W, W((x^{-1}))), \quad (2.11)$$

where by definition

$$Y^o(v, x + z_0)w = (Y^o(v, y)w)|_{y=x+z_0} \quad (2.12)$$

for  $w \in W$ . Let  $U$  be a vector space, e.g.,  $U = \mathbb{C}$ . For  $v \in V$ ,  $f \in \text{Hom}(W, U)$ ,  $z_0 \in \mathbb{C}$ , the compositions  $fY^o(v, x)$  and  $fY^o(v, x + z_0)$  are elements of  $(\text{Hom}(W, U))[[x, x^{-1}]]$ .

We recall the following result from [Li2]:

**Proposition 2.2** *Let  $(W, Y_W)$  be a right weak  $V$ -module and let  $z_0$  be any complex number. For  $v \in V$ , we set*

$$Y_W^{(z_0)}(v, x) = Y_W(v, x + z_0) = e^{z_0 \frac{d}{dx}} Y_W(v, x) \in (\text{End } W)[[x, x^{-1}]]. \quad (2.13)$$

*Then the pair  $(W, Y_W^{(z_0)})$  is also a right weak  $V$ -module.*

Next we briefly recall the construction of the weak  $V \otimes V$ -module  $\mathcal{D}_{P(z)}(W)$  from [Li2].

**Definition 2.3** Let  $W$  be a weak  $V$ -module and  $z$  a nonzero complex number. A  $P(z)$ -linear functional on  $W$  is a linear functional  $\alpha$  on  $W$  that satisfies the following condition: For each  $v \in V$ , there exist  $k, l \in \mathbb{N}$  such that

$$x^l (x - z)^k \langle \alpha, Y^o(v, x) w \rangle \in \mathbb{C}[x] \quad (2.14)$$

for all  $w \in W$ , or what is equivalent to, the series  $\langle \alpha, Y^o(v, x) w \rangle$ , an element of  $\mathbb{C}((x^{-1}))$ , absolutely converges in the domain  $|x| > |z|$  to a rational function of the form

$$x^{-l} (x - z)^{-k} g(x), \quad (2.15)$$

where  $g(x) \in \mathbb{C}[x]$ .

All  $P(z)$ -linear functionals on  $W$  form a subspace of  $W^*$ , denoted by  $\mathcal{D}_{P(z)}(W)$ .

The following is an obvious characterization of a  $P(z)$ -linear functional without involving matrix-coefficients.

**Lemma 2.4** [Li2] *Let  $W, z$  be given as before and let  $\alpha \in W^*$ . Then  $\alpha \in \mathcal{D}_{P(z)}(W)$  if and only if for  $v \in V$ , there exist  $k, l \in \mathbb{N}$  such that*

$$x^l (x - z)^k \alpha Y^o(v, x) \in W^*[[x]], \quad (2.16)$$

*or equivalently, if and only if for  $v \in V$ , there exists  $k \in \mathbb{N}$  such that*

$$(x - z)^k \alpha Y^o(v, x) \in W^*((x)). \quad (2.17)$$

Denote by  $\mathbb{C}(x)$  the algebra of rational functions of  $x$ . The  $\iota$ -maps  $\iota_{x;0}$  and  $\iota_{x;\infty}$  from  $\mathbb{C}(x)$  to  $\mathbb{C}[[x, x^{-1}]]$  were defined as follows: For any rational function  $f(x)$ ,  $\iota_{x;0}f(x)$  is the Laurent series expansion of  $f(x)$  at  $x = 0$  and  $\iota_{x;\infty}f(x)$  is the Laurent series expansion of  $f(x)$  at  $x = \infty$ . These are injective  $\mathbb{C}[x, x^{-1}]$ -linear maps. Using the formal variable convention, we have

$$\iota_{x;0}((x - z)^n f(x)) = (-z + x)^n \iota_{x;0}f(x), \quad (2.18)$$

$$\iota_{x;\infty}((x - z)^n f(x)) = (x - z)^n \iota_{x;\infty}f(x) \quad (2.19)$$

for  $n \in \mathbb{Z}$ ,  $z \in \mathbb{C}^\times$ ,  $f(x) \in \mathbb{C}(x)$ . Furthermore, for  $z \in \mathbb{C}$ ,  $f(x) \in \mathbb{C}(x)$ ,

$$\iota_{x;0}f(x) = (\iota_{y;\infty}f(y^{-1}))|_{y=x^{-1}}, \quad (2.20)$$

$$\iota_{x;\infty}f(x - z) = (\iota_{y;\infty}f(y))|_{y=x-z}. \quad (2.21)$$

Note that  $\iota_{y;\infty}f(y) \in \mathbb{C}((y^{-1}))$ , so that  $(\iota_{y;\infty}f(y))|_{y=x-z}$  exists in  $\mathbb{C}[[x, x^{-1}]]$ .

From the definition, for  $\alpha \in \mathcal{D}_{P(z)}(W)$ ,  $v \in V$ ,  $w \in W$ ,  $\langle \alpha, Y^o(v, x)w \rangle$  lies in the range of  $\iota_{x;\infty}$ . Then  $\iota_{x;\infty}^{-1}\langle \alpha, Y^o(v, x)w \rangle$  is a well defined element of  $\mathbb{C}(x)$ . It is also true that  $\langle \alpha, Y^o(v, x+z)w \rangle$  lies in the range of  $\iota_{x;\infty}$ .

**Definition 2.5** [Li2] For  $v \in V$ ,  $\alpha \in \mathcal{D}_{P(z)}(W)$ , we define

$$Y_{P(z)}^L(v, x)\alpha, \quad Y_{P(z)}^R(v, x)\alpha \in W^*[[x, x^{-1}]]$$

by

$$\langle Y_{P(z)}^L(v, x)\alpha, w \rangle = \iota_{x;0} \left( \iota_{x;\infty}^{-1} \langle \alpha, Y^o(v, x+z)w \rangle \right) \quad (2.22)$$

$$\langle Y_{P(z)}^R(v, x)\alpha, w \rangle = \iota_{x;0} \iota_{x;\infty}^{-1} \langle \alpha, Y^o(v, x)w \rangle \quad (2.23)$$

for  $w \in W$ .

We have ([Li2], Proposition 3.22):

**Proposition 2.6** Let  $v \in V$ ,  $\alpha \in \mathcal{D}_{P(z)}(W)$ . Then

$$\begin{aligned} & x_0^{-1} \delta \left( \frac{x-z}{x_0} \right) \alpha Y^o(v, x) - x_0^{-1} \delta \left( \frac{z-x}{-x_0} \right) Y_{P(z)}^R(v, x)\alpha \\ &= z^{-1} \delta \left( \frac{x-x_0}{z} \right) Y_{P(z)}^L(v, x_0)\alpha. \end{aligned} \quad (2.24)$$

As an immediate consequence we have:

**Corollary 2.7** [Li2] Let  $v \in V$ ,  $\alpha \in \mathcal{D}_{P(z)}(W)$ . Then

$$(-z+x)^k Y_{P(z)}^R(v, x)\alpha = (x-z)^k \alpha Y^o(v, x), \quad (2.25)$$

$$(z+x)^l Y_{P(z)}^L(v, x)\alpha = (x+z)^l \alpha Y^o(v, x+z), \quad (2.26)$$

where  $k$  and  $l$  are any pair of integers such that (2.16) holds. (The integers  $k$  and  $l$  could be negative.)

We have ([Li2], Proposition 3.24):

**Proposition 2.8** Let  $W$  and  $z$  be given as before. Then

$$Y_{P(z)}^L(v, x)\alpha, \quad Y_{P(z)}^R(v, x)\alpha \in (\mathcal{D}_{P(z)}(W))((x)) \quad (2.27)$$

for  $v \in V$ ,  $\alpha \in \mathcal{D}_{P(z)}(W)$ . Furthermore,

$$Y_{P(z)}^L(u, x_1) Y_{P(z)}^R(v, x_2) = Y_{P(z)}^R(v, x_2) Y_{P(z)}^L(u, x_1) \quad (2.28)$$

on  $\mathcal{D}_{P(z)}(W)$  for  $u, v \in V$ .

In view of Proposition 2.8,  $Y_{P(z)}^L$  and  $Y_{P(z)}^R$  give rise to a well defined linear map

$$Y_{P(z)} = Y_{P(z)}^L \otimes Y_{P(z)}^R : V \otimes V \rightarrow \left( \text{End } \mathcal{D}_{P(z)}(W) \right) [[x, x^{-1}]]. \quad (2.29)$$

Then we have ([Li2], Theorem 3.17, Propositions 3.21 and 3.24 and Theorem 3.25):

**Theorem 2.9** *Let  $W$  be a weak  $V$ -module and  $z$  a nonzero complex number. Then both the pairs  $(\mathcal{D}_{P(z)}(W), Y_{P(z)}^L)$  and  $(\mathcal{D}_{P(z)}(W), Y_{P(z)}^R)$  carry the structure of a weak  $V \otimes V$ -module and the pair  $(\mathcal{D}_{P(z)}(W), Y_{P(z)})$  carries the structure of a weak  $V \otimes V$ -module.*

As pointed out in [Li2],  $\mathcal{D}_{P(z)} : W \mapsto \mathcal{D}_{P(z)}(W)$  is a contravariant functor from the category of weak  $V$ -modules to the category of weak  $V \otimes V$ -modules. For a general  $V$ , even if  $W$  is a  $V$ -module,  $\mathcal{D}_{P(z)}(W)$  may be not an (ordinary)  $V \otimes V$ -module. Recall from [Li2] that  $R_{P(z)}(W)$  is the sum of  $V \otimes V$ -submodules of  $\mathcal{D}_{P(z)}(W)$ , on which  $L(0) \otimes 1$  and  $1 \otimes L(0)$  act semisimply. Denote by  $\mathcal{C}_{V \otimes V}^o$  the category of  $V \otimes V$ -modules on which  $L(0) \otimes 1$  and  $1 \otimes L(0)$  act semisimply.

Vertex operator algebra  $V$  is said to be *rational in the sense of Huang-Lepowsky* if  $V$  has only finitely many irreducible modules up to equivalence, every  $V$ -module is completely reducible and if the fusion rule for every triple of irreducible  $V$ -modules is finite. If  $V$  is rational in the sense of Huang-Lepowsky, it was proved in [Li2] (Theorem 4.7) that for any  $V$ -module  $W$ ,  $R_{P(z)}(W)$  is a  $V \otimes V$ -module. Then  $\mathcal{F}_{P(z)} : W \mapsto R_{P(z)}(W)$  is a coinvariant functor from the category  $\mathcal{C}_V$  of  $V$ -modules to the category  $\mathcal{C}_{V \otimes V}^o$ .

### 3 Weak $V \otimes V$ -module $\mathcal{D}_{Q(z)}(W)$

In this section, we shall construct a weak  $V \otimes V$ -module  $\mathcal{D}_{Q(z)}(W)$  for any weak  $V$ -module  $W$ .

**Definition 3.1** Let  $W$  be a (left) weak  $V$ -module and  $z$  a nonzero complex number. A linear functional  $\alpha$  on  $W$  is called a  $Q(z)$ -linear functional if for  $v \in V$ ,  $w \in W$ , the formal series

$$\langle \alpha, Y(v, x)w \rangle \left( = \sum_{n \in \mathbb{Z}} \langle \alpha, v_n w \rangle x^{-n-1} \right),$$

an element of  $\mathbb{C}((x))$ , absolutely converges in the domain  $0 < |x| < |z|$  to a rational function  $f_\alpha(v, w; x) \in \mathbb{C}[x, x^{-1}, (x+z)^{-1}]$  such that when  $v$  is fixed with  $w$  being free, the orders of the possible poles at  $x = -z, \infty$  of  $f_\alpha(v, w; x)$  being viewed as meromorphic functions on the sphere  $\mathbb{C} \cup \{\infty\}$  are uniformly bounded.

All  $Q(z)$ -linear functionals on  $W$  clearly form a subspace of  $W^*$ , which we denote by  $\mathcal{D}_{Q(z)}(W)$ . Let  $\alpha \in \mathcal{D}_{Q(z)}(W)$ . From the definition, for  $v \in V$ , there exist nonnegative integers  $m$  and  $l$  such that

$$x^{-m}(x+z)^l \langle \alpha, Y(v, x)w \rangle \in \mathbb{C}[x^{-1}] \quad (3.1)$$

for all  $w \in W$ . (Note that  $m$  depends on  $l$ .) Thus

$$x^{-m}(x+z)^l \alpha Y(v, x) \in W^*[[x^{-1}]], \quad (3.2)$$

or equivalently,

$$(x+z)^l \alpha Y(v, x) \in W^*((x^{-1})). \quad (3.3)$$

Conversely, if  $\alpha \in W^*$  satisfies the condition that for every  $v \in V$ , there exists a nonnegative integer  $l$  such that (3.3) holds, then we easily see that  $\alpha \in \mathcal{D}_{Q(z)}(W)$ . (From (3.2) to (3.1) we also use the truncation condition:  $Y(v, x)w \in W((x))$ .) This gives a slightly different characterization of a  $Q(z)$ -linear functional.

**Lemma 3.2** *Let  $\alpha \in W^*$ . Then  $\alpha \in \mathcal{D}_{Q(z)}(W)$  if and only if for every  $v \in V$ , there exists a nonnegative integer  $l$  such that (3.3) holds.  $\square$*

In the following we shall identify  $\mathcal{D}_{Q(z)}(W)$  with  $\mathcal{D}_{P(-z^{-1})}(W)$ .

**Definition 3.3** Let  $W$  and  $z$  be given as before. For  $v \in V$ ,  $\alpha \in \mathcal{D}_{Q(z)}(W)$ , we define

$$\tilde{Y}_{Q(z)}(v, x)\alpha \in W^*[[x, x^{-1}]] \quad (3.4)$$

by

$$\langle \tilde{Y}_{Q(z)}(v, x)\alpha, w \rangle = \iota_{x; \infty} \iota_{x; 0}^{-1} \langle \alpha, Y(v, x)w \rangle \quad (3.5)$$

for  $w \in W$ .

Let  $v \in V$ ,  $\alpha \in \mathcal{D}_{Q(z)}(W)$  and let  $l \in \mathbb{N}$  be such that (3.3) holds. Then

$$(x+z)^l \langle \alpha, Y(v, x)w \rangle \in \mathbb{C}[x, x^{-1}]$$

for  $w \in W$ . Since  $\iota$ -maps are  $\mathbb{C}[x, x^{-1}]$ -linear and map 1 to 1, from (3.5) we get

$$(x+z)^l \langle \tilde{Y}_{Q(z)}(v, x)\alpha, w \rangle = (x+z)^l \langle \alpha, Y(v, x)w \rangle. \quad (3.6)$$

Thus

$$(x+z)^l \tilde{Y}_{Q(z)}(v, x)\alpha = (x+z)^l \alpha Y(v, x). \quad (3.7)$$

**Proposition 3.4** *Let  $W$  and  $z$  be given as before. Then*

$$\mathcal{D}_{Q(z)}(W) = \mathcal{D}_{P(-z^{-1})}(W). \quad (3.8)$$

Furthermore,

$$\tilde{Y}_{Q(z)}^o(v, x)\alpha = Y_{P(-z^{-1})}^R(v, x)\alpha \quad (3.9)$$

for  $v \in V$ ,  $\alpha \in \mathcal{D}_{Q(z)}(W)$ .



**Proof.** Let  $\alpha \in W^*$ . From Definition 3.1,  $\alpha \in \mathcal{D}_{Q(z)}(W)$  if and only if for  $v \in V$ ,  $w \in W$ , the formal series

$$\langle \alpha, Y(v, x^{-1})w \rangle$$

converges in the domain  $|x| > |z^{-1}|$  to a rational function in  $\mathbb{C}[x, x^{-1}, (x+z^{-1})^{-1}]$  such that when  $v$  is fixed with  $w$  being free, the orders of possible poles at 0 and  $-z^{-1}$  are uniformly bounded. Furthermore, since  $e^{xL(1)}(-x^{-2})^{L(0)}v \in V[x, x^{-1}]$  for  $v \in V$ ,  $\alpha \in \mathcal{D}_{Q(z)}(W)$  if and only if for  $v \in V$ ,  $w \in W$ , the formal series  $\langle \alpha, Y^o(v, x)w \rangle$  absolutely converges in the domain  $|x| > |z^{-1}|$  to a rational function in  $\mathbb{C}[x, x^{-1}, (x+z^{-1})^{-1}]$  such that when  $v$  is fixed with  $w$  being free, the orders of possible poles at 0 and  $-z^{-1}$  are uniformly bounded. Thus,  $\alpha$  is a  $Q(z)$ -linear functional if and only if  $\alpha$  is a  $P(-z^{-1})$ -linear functional. This proves the first part of the proposition.

For the second part, from (3.5) (also using (2.20)) we have

$$\langle \tilde{Y}_{Q(z)}(v, x^{-1})\alpha, w \rangle = \iota_{x;0}\iota_{x;\infty}^{-1} \langle \alpha, Y(v, x^{-1})w \rangle. \quad (3.10)$$

Since  $e^{xL(1)}(-x^{-2})^{L(0)}v \in V[x, x^{-1}]$  for  $v \in V$ , and the  $\iota$ -maps are  $\mathbb{C}[x, x^{-1}]$ -linear and send 1 to 1, we have

$$\langle \tilde{Y}_{Q(z)}^o(v, x)\alpha, w \rangle = \iota_{x;0}\iota_{x;\infty}^{-1} \langle \alpha, Y^o(v, x)w \rangle. \quad (3.11)$$

Then (3.9) follows immediately from the definition of  $Y_{P(-z^{-1})}^R$ .  $\square$

Because  $(\mathcal{D}_{P(-z^{-1})}(W), Y_{P(-z^{-1})}^R)$  carries the structure of a left weak  $V$ -module (Theorem 2.9), in view of Propositions 3.4 and 2.1 we immediately have (cf. [Li2], Theorem 3.17):

**Theorem 3.5** *Let  $W$  be a (left) weak  $V$ -module and let  $z$  be a nonzero complex number. Then the pair  $(\mathcal{D}_{Q(z)}(W), \tilde{Y}_{Q(z)})$  carries the structure of a right weak  $V$ -module.  $\square$*

Next, we are going to use  $\tilde{Y}_{Q(z)}$  to define two left module structures on  $\mathcal{D}_{Q(z)}(W)$ .

**Definition 3.6** For  $v \in V$ ,  $\alpha \in \mathcal{D}_{Q(z)}(W)$ , we define

$$Y_{Q(z)}^R(v, x)\alpha, \quad Y_{Q(z)}^L(v, x)\alpha \in W^*[[x, x^{-1}]] \quad (3.12)$$

by

$$\langle (Y_{Q(z)}^L)^o(v, x)\alpha, w \rangle = \iota_{x;\infty} \left( \iota_{y;0}^{-1} \langle \alpha, Y(v, y)w \rangle \right) \Big|_{y=x-z}, \quad (3.13)$$

$$\langle Y_{Q(z)}^R(v, x)\alpha, w \rangle = \iota_{x;0} \left( \iota_{y;0}^{-1} \langle \alpha, Y(v, y)w \rangle \right) \Big|_{y=x-z} \quad (3.14)$$

for  $w \in W$ .

Let  $v \in V$ ,  $\alpha \in \mathcal{D}_{Q(z)}(W)$ ,  $w \in W$ . Using (3.13) and (3.5), we get

$$\begin{aligned} \langle (Y_{Q(z)}^L)^o(v, x)\alpha, w \rangle &= \left( \iota_{y;\infty} \iota_{y;0}^{-1} \langle \alpha, Y(v, y)w \rangle \right) \Big|_{y=x-z} \\ &= \left( \langle \tilde{Y}_{Q(z)}(v, y)\alpha, w \rangle \right) \Big|_{y=x-z} \\ &= \langle \tilde{Y}_{Q(z)}(v, x-z)\alpha, w \rangle \\ &= \langle \tilde{Y}_{Q(z)}^{(-z)}(v, x)\alpha, w \rangle. \end{aligned} \quad (3.15)$$

Notice that  $\langle \tilde{Y}_{Q(z)}(v, x-z)\alpha, w \rangle$  exists because  $\tilde{Y}_{Q(z)}(v, y)\alpha \in \mathcal{D}_{Q(z)}(W)((y^{-1}))$ . (But  $\langle \alpha, Y(v, x-z)w \rangle$  does not exist.) Thus

$$(Y_{Q(z)}^L)^o(v, x)\alpha = \tilde{Y}_{Q(z)}^{(-z)}(v, x)\alpha. \quad (3.16)$$

Similarly, using (3.14) and (3.5) (also using (2.21)) we get

$$\begin{aligned} \langle Y_{Q(z)}^R(v, x)\alpha, w \rangle &= \iota_{x;0} \left( \iota_{y;\infty}^{-1} \langle \tilde{Y}_{Q(z)}(v, y)\alpha, w \rangle \right) |_{y=x-z} \\ &= \iota_{x;0} \iota_{x;\infty}^{-1} \langle \tilde{Y}_{Q(z)}(v, x-z)\alpha, w \rangle \\ &= \iota_{x;0} \iota_{x;\infty}^{-1} \langle \tilde{Y}_{Q(z)}^{(-z)}(v, x)\alpha, w \rangle. \end{aligned} \quad (3.17)$$

That is,

$$\iota_{x;0}^{-1} \langle Y_{Q(z)}^R(v, x)\alpha, w \rangle = \iota_{x;\infty}^{-1} \langle \tilde{Y}_{Q(z)}^{(-z)}(v, x)\alpha, w \rangle. \quad (3.18)$$

Notice that

$$\iota_{x;0}^{-1} \langle \alpha, Y(v, x)w \rangle = \iota_{x;\infty}^{-1} \langle \tilde{Y}_{Q(z)}(v, x)\alpha, w \rangle = \left( \iota_{y;\infty}^{-1} \langle \tilde{Y}_{Q(z)}^{(-z)}(v, y)\alpha, w \rangle \right) |_{y=x+z}. \quad (3.19)$$

Combining (3.18) with (3.19), then using the fundamental properties of the delta function ([FHL], Proposition 3.1.1.) we get

$$\begin{aligned} &x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) \langle \tilde{Y}_{Q(z)}^{(-z)}(v, x_1)\alpha, w \rangle - x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) \langle Y_{Q(z)}^R(v, x_1)\alpha, w \rangle \\ &= z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) \langle \alpha, Y(v, x_0)w \rangle. \end{aligned} \quad (3.20)$$

By using the delta-function substitution property, dropping out  $w$ , we get another version:

$$\begin{aligned} &x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) \tilde{Y}_{Q(z)}(v, x_0)\alpha - x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) Y_{Q(z)}^R(v, x_1)\alpha \\ &= z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) \alpha Y(v, x_0). \end{aligned} \quad (3.21)$$

Then we immediately have:

**Proposition 3.7** *For  $v \in V$ ,  $\alpha \in \mathcal{D}_{Q(z)}(W)$ ,*

$$\begin{aligned} &z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) \alpha Y(v, x_0) \\ &= x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) (Y_{Q(z)}^L)^o(v, x_1)\alpha - x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) Y_{Q(z)}^R(v, x_1)\alpha. \quad \square \end{aligned} \quad (3.22)$$

**Proposition 3.8** *Let  $W$  be a (left) weak  $V$ -module and let  $z$  be a nonzero complex number. Then the pair  $(\mathcal{D}_{Q(z)}(W), Y_{Q(z)}^L)$  carries the structure of a (left) weak  $V$ -module.*

**Proof.** Because  $\tilde{Y}_{Q(z)}$  gives rise to a right weak  $V$ -module structure on  $\mathcal{D}_{Q(z)}(W)$  (Theorem 3.5), by Proposition 2.2,  $\tilde{Y}_{Q(z)}^{(-z)}$  also gives rise to a right weak  $V$ -module structure on  $\mathcal{D}_{Q(z)}(W)$ . Furthermore, by Proposition 2.1,  $Y_{Q(z)}^L$  gives rise to a left weak  $V$ -module structure on  $\mathcal{D}_{Q(z)}(W)$  because  $(Y_{Q(z)}^L)^o = \tilde{Y}_{Q(z)}^{(-z)}$ .  $\square$

We shall prove that the pair  $(\mathcal{D}_{Q(z)}(W), Y_{Q(z)}^R)$  also carries the structure of a left weak  $V$ -module. First we have:

**Lemma 3.9** *Let  $W, z$  be given as before. Then for  $v \in V$ ,  $\alpha \in \mathcal{D}_{Q(z)}(W)$ ,  $n \in \mathbb{Z}$ ,*

$$(z+x)^n \alpha Y(v, x) \in \mathcal{D}_{Q(z)}(W)[[x, x^{-1}]], \quad (3.23)$$

$$Y_{Q(z)}^R(v, x) \alpha \in \mathcal{D}_{Q(z)}(W)((x)). \quad (3.24)$$

**Proof.** Let  $u \in V$ ,  $m \in \mathbb{Z}$ . From Borchers' commutator formula we have

$$\alpha v_m Y(u, x) = \alpha Y(u, x) v_m - \sum_{i \geq 0} \binom{m}{i} x^{m-i} \alpha Y(v_i u, x). \quad (3.25)$$

Since  $v_i u \neq 0$  for only finitely many  $i \geq 0$ , there exists  $l \in \mathbb{N}$  such that

$$(x+z)^l \alpha Y(u, x), \quad (x+z)^l \alpha Y(v_i u, x) \in W^*((x^{-1})) \quad (3.26)$$

for all  $i \geq 0$ . Then using (3.25) we get

$$(x+z)^l \alpha v_m Y(u, x) \in W^*((x^{-1})) \quad (3.27)$$

for all  $m \in \mathbb{Z}$ . Thus  $\alpha v_m \in \mathcal{D}_{Q(z)}(W)$ . Notice that for  $n \in \mathbb{Z}$ ,

$$(z+x_0)^n \alpha Y(v, x_0) \text{ exists}$$

in  $W^*[[x_0, x_0^{-1}]]$  though its coefficient of  $x_0^t$  for  $t \in \mathbb{Z}$  is generally an infinite sum of  $\alpha v_m$ 's. Furthermore, from (3.27) we have

$$(x+z)^l (z+x_0)^n \alpha Y(v, x_0) Y(u, x) \in (W^*[[x_0, x_0^{-1}]])((x^{-1})). \quad (3.28)$$

Then (3.23) follows from Lemma 3.2.

Let  $l \in \mathbb{N}$  be such that (3.7) holds. Taking  $\text{Res}_{x_0}$  from (3.21), using the fundamental properties of delta functions, and using (3.7) and (3.23) we get

$$\begin{aligned} & Y_{Q(z)}^R(v, x_1) \alpha \\ &= \text{Res}_{x_0} x_1^{-1} \delta\left(\frac{x_0+z}{x_1}\right) \tilde{Y}_{Q(z)}(v, x_0) \alpha - \text{Res}_{x_0} x_1^{-1} \delta\left(\frac{z+x_0}{x_1}\right) \alpha Y(v, x_0) \\ &= \text{Res}_{x_0} \sum_{n \in \mathbb{Z}} x_1^{-n-1} \left( (x_0+z)^n \tilde{Y}_{Q(z)}(v, x_0) \alpha - (z+x_0)^n \alpha Y(v, x_0) \right) \\ &= \text{Res}_{x_0} \sum_{n < l} x_1^{-n-1} \left( (x_0+z)^n \tilde{Y}_{Q(z)}(v, x_0) \alpha - (z+x_0)^n \alpha Y(v, x_0) \right). \end{aligned} \quad (3.29)$$

Noticing that for any  $n \in \mathbb{Z}$ , every coefficient of monomials in  $x$  of  $(x+z)^n \tilde{Y}_{Q(z)}(v, x)\alpha$  is a finite sum of the coefficients of  $\tilde{Y}_{Q(z)}(v, x)\alpha$ , with Theorem 3.5 we get

$$(x+z)^n \tilde{Y}_{Q(z)}(v, x)\alpha \in \mathcal{D}_{Q(z)}(W)((x^{-1})).$$

Then (3.24) immediately follows from (3.23) and (3.29).  $\square$

We have the following commutativity relations:

**Proposition 3.10** *Let  $W$  be a (left) weak  $V$ -module and let  $z$  be a nonzero complex number. Then*

$$\tilde{Y}_{Q(z)}(u, x_1)Y_{Q(z)}^R(v, x_2) = Y_{Q(z)}^R(v, x_2)\tilde{Y}_{Q(z)}(u, x_1), \quad (3.30)$$

$$Y_{Q(z)}^L(u, x_1)Y_{Q(z)}^R(v, x_2) = Y_{Q(z)}^R(v, x_2)Y_{Q(z)}^L(u, x_1) \quad (3.31)$$

on  $\mathcal{D}_{Q(z)}(W)$  for  $u, v \in V$ .

**Proof.** In view of the connection between  $Y_{Q(z)}^L(u, x)$  and  $\tilde{Y}_{Q(z)}(v, x)$  (recall (3.16)), we only need to prove (3.30). The proof is similar to that of Proposition 3.24 of [Li2].

It follows (cf. [DL]) from the (opposite) Jacobi identity that there exists  $k \in \mathbb{N}$  such that

$$(x_0 - x_2)^k \alpha Y(u, x_2)Y(v, x_0) = (x_0 - x_2)^k \alpha Y(v, x_0)Y(u, x_2), \quad (3.32)$$

$$(x_0 - x_2)^k \tilde{Y}_{Q(z)}(u, x_2)\tilde{Y}_{Q(z)}(v, x_0)\alpha = (x_0 - x_2)^k \tilde{Y}_{Q(z)}(v, x_0)\tilde{Y}_{Q(z)}(u, x_2)\alpha. \quad (3.33)$$

In view of (3.27) and (3.7) there exists  $l \in \mathbb{N}$  such that

$$(x_2 + z)^l \tilde{Y}_{Q(z)}(u, x_2)\alpha = (x_2 + z)^l \alpha Y(u, x_2), \quad (3.34)$$

$$(x_2 + z)^l \tilde{Y}_{Q(z)}(u, x_2)(\alpha Y(v, x_0)) = (x_2 + z)^l \alpha Y(v, x_0)Y(u, x_2). \quad (3.35)$$

Using (3.21) together with all the identities above we have

$$\begin{aligned} & x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) (x_0 - x_2)^k (x_2 + z)^l \tilde{Y}_{Q(z)}(u, x_2) Y_{Q(z)}^R(v, x_1) \alpha \\ = & x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) (x_0 - x_2)^k (x_2 + z)^l \tilde{Y}_{Q(z)}(u, x_2) \tilde{Y}_{Q(z)}(v, x_0) \alpha \\ & - z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) (x_0 - x_2)^k (x_2 + z)^l \tilde{Y}_{Q(z)}(u, x_2) (\alpha Y(v, x_0)) \\ = & x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) (x_0 - x_2)^k (x_2 + z)^l \tilde{Y}_{Q(z)}(u, x_2) \tilde{Y}_{Q(z)}(v, x_0) \alpha \\ & - z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) (x_0 - x_2)^k (x_2 + z)^l \alpha Y(v, x_0) Y(u, x_2) \\ = & x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) (x_0 - x_2)^k (x_2 + z)^l \tilde{Y}_{Q(z)}(v, x_0) \tilde{Y}_{Q(z)}(u, x_2) \alpha \\ & - z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) (x_0 - x_2)^k (x_2 + z)^l \alpha Y(u, x_2) Y(v, x_0) \end{aligned}$$

$$\begin{aligned}
&= x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) (x_0 - x_2)^k (x_2 + z)^l \tilde{Y}_{Q(z)}(v, x_0) \tilde{Y}_{Q(z)}(u, x_2) \alpha \\
&\quad - z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) (x_0 - x_2)^k (x_2 + z)^l (\tilde{Y}_{Q(z)}(u, x_2) \alpha) Y(v, x_0) \\
&= x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) (x_0 - x_2)^k (x_2 + z)^l Y_{Q(z)}^R(v, x_1) \tilde{Y}_{Q(z)}(u, x_2) \alpha.
\end{aligned} \tag{3.36}$$

By taking  $\text{Res}_{x_0}$  and using the fundamental properties of delta functions we get

$$\begin{aligned}
&(-z + x_1 - x_2)^k (x_2 + z)^l \tilde{Y}_{Q(z)}(u, x_2) Y_{Q(z)}^R(v, x_1) \alpha \\
&= (-z + x_1 - x_2)^k (x_2 + z)^l Y_{Q(z)}^R(v, x_1) \tilde{Y}_{Q(z)}(u, x_2) \alpha.
\end{aligned} \tag{3.37}$$

Then multiplying both sides by  $(-z + x_1 - x_2)^{-k} (x_2 + z)^{-l}$  we obtain (3.30), noting that we are allowed to multiply.  $\square$

Now we are ready to prove:

**Proposition 3.11** *Let  $W$  be a (left) weak  $V$ -module and let  $z$  be a nonzero complex number. Then the pair  $(\mathcal{D}_{Q(z)}(W), Y_{Q(z)}^R)$  carries the structure of a (left) weak  $V$ -module.*

**Proof.** It is clear that  $Y_{Q(z)}^R(\mathbf{1}, x) = 1$  because  $\tilde{Y}_{Q(z)}(\mathbf{1}, x) = 1$ . With Lemma 3.9, we only need to prove the Jacobi identity.

For convenience, let us locally use  $\bar{Y}$  for  $\tilde{Y}_{Q(z)}^{(-z)}$ . That is,  $(\mathcal{D}_{Q(z)}(W), \bar{Y})$  is a right weak  $V$ -module. Furthermore, for  $v \in V$ ,  $w \in W$ , if  $l$  is a nonnegative integer such that  $x^l Y(v, x) w \in W[[x]]$ , then by taking  $\text{Res}_{x_0} x_0^l$  from (3.20) we get

$$(x - z)^l \langle Y_{Q(z)}^R(v, x) \alpha, w \rangle = (x - z)^l \langle \bar{Y}(v, x) \alpha, w \rangle \tag{3.38}$$

for all  $\alpha \in \mathcal{D}_{Q(z)}(W)$ . Furthermore, from (3.30)  $\bar{Y}$  and  $R_{Q(z)}^R$  commute.

Now, let  $u, v \in V$ ,  $\alpha \in \mathcal{D}_{Q(z)}(W)$ ,  $w \in W$ . Let  $l \in \mathbb{N}$  be such that

$$x^l Y(u, x) w, \quad x^l Y(v, x) w \in W[[x]]. \tag{3.39}$$

Then

$$(x_1 - z)^l \langle Y_{Q(z)}^R(u, x_1) Y_{Q(z)}^R(v, x_2) \alpha, w \rangle = (x_1 - z)^l \langle \bar{Y}(u, x_1) Y_{Q(z)}^R(v, x_2) \alpha, w \rangle, \tag{3.40}$$

$$(x_2 - z)^l \langle Y_{Q(z)}^R(v, x_2) \tilde{Y}(u, x_1) \alpha, w \rangle = (x_2 - z)^l \langle \bar{Y}(v, x_2) \tilde{Y}(u, x_1) \alpha, w \rangle \tag{3.41}$$

because

$$Y_{Q(z)}^R(v, x) \alpha, \quad \bar{Y}(u, x) \alpha \in \mathcal{D}_{Q(z)}(W)[[x, x^{-1}]].$$

Then

$$\begin{aligned}
&(x_1 - z)^l (x_2 - z)^l \langle Y_{Q(z)}^R(u, x_1) Y_{Q(z)}^R(v, x_2) \alpha, w \rangle \\
&= (x_1 - z)^l (x_2 - z)^l \langle \bar{Y}(u, x_1) Y_{Q(z)}^R(v, x_2) \alpha, w \rangle \\
&= (x_1 - z)^l (x_2 - z)^l \langle Y_{Q(z)}^R(v, x_2) \bar{Y}(u, x_1) \alpha, w \rangle \\
&= (x_1 - z)^l (x_2 - z)^l \langle \bar{Y}(v, x_2) \bar{Y}(u, x_1) \alpha, w \rangle.
\end{aligned} \tag{3.42}$$

Similarly we have

$$\begin{aligned} & (x_1 - z)^l (x_2 - z)^l \langle Y_{Q(z)}^R(v, x_2) Y_{Q(z)}^R(u, x_1) \alpha, w \rangle \\ &= (x_1 - z)^l (x_2 - z)^l \langle \bar{Y}(u, x_1) \bar{Y}(v, x_2) \alpha, w \rangle. \end{aligned} \quad (3.43)$$

Using (3.42), (3.43) and the opposite Jacobi identity for  $\bar{Y}$  we get

$$\begin{aligned} & x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) (x_1 - z)^l (x_2 - z)^l \langle Y_{Q(z)}^R(u, x_1) Y_{Q(z)}^R(v, x_2) \alpha, w \rangle \\ & - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) (x_1 - z)^l (x_2 - z)^l \langle Y_{Q(z)}^R(v, x_2) Y_{Q(z)}^R(u, x_1) \alpha, w \rangle \\ &= x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) (x_1 - z)^l (x_2 - z)^l \langle \bar{Y}(Y(u, x_0)v, x_2) \alpha, w \rangle. \end{aligned} \quad (3.44)$$

Let  $m$  be any fixed integer. Since only finitely many  $u_n v$  are nonzero for  $n \geq m$ , there exists  $l' \in \mathbb{N}$  (depending on  $m$ ) such that

$$(x_2 - z)^{l'} \langle Y_{Q(z)}^R(u_n v, x_2) \alpha, w \rangle = (x_2 - z)^{l'} \langle \bar{Y}(u_n v, x_2) \alpha, w \rangle \quad (3.45)$$

for all  $n \geq m$ . Then

$$\begin{aligned} & \text{Res}_{x_0} x_0^m x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) (x_2 - z)^{l'} \langle \bar{Y}(Y(u, x_0)v, x_2) \alpha, w \rangle \\ &= \text{Res}_{x_0} x_0^m x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) (x_2 - z)^{l'} \langle Y_{Q(z)}^R(Y(u, x_0)v, x_2) \alpha, w \rangle. \end{aligned} \quad (3.46)$$

Let  $l'' = l + l'$ . Combining (3.44) with (3.46) we get

$$\begin{aligned} & \text{Res}_{x_0} x_0^m x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) (x_1 - z)^{l''} (x_2 - z)^{l''} \langle Y_{Q(z)}^R(u, x_1) Y_{Q(z)}^R(v, x_2) \alpha, w \rangle \\ & - \text{Res}_{x_0} x_0^m x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) (x_1 - z)^{l''} (x_2 - z)^{l''} \langle Y_{Q(z)}^R(v, x_2) Y_{Q(z)}^R(u, x_1) \alpha, w \rangle \\ &= \text{Res}_{x_0} x_0^m x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) (x_1 - z)^{l''} (x_2 - z)^{l''} \langle Y_{Q(z)}^R(Y(u, x_0)v, x_2) \alpha, w \rangle. \end{aligned} \quad (3.47)$$

Multiplying (3.47) by  $(-z + x_1)^{-l''} (-z + x_2)^{-l''}$  we obtain

$$\begin{aligned} & \text{Res}_{x_0} x_0^m x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) \langle Y_{Q(z)}^R(u, x_1) Y_{Q(z)}^R(v, x_2) \alpha, w \rangle \\ & - \text{Res}_{x_0} x_0^m x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) \langle Y_{Q(z)}^R(v, x_2) Y_{Q(z)}^R(u, x_1) \alpha, w \rangle \\ &= \text{Res}_{x_0} x_0^m x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) \langle Y_{Q(z)}^R(Y(u, x_0)v, x_2) \alpha, w \rangle. \end{aligned} \quad (3.48)$$

Since  $m$  was arbitrarily fixed, we may drop off  $\text{Res}_{x_0} x_0^m$  to obtain the Jacobi identity for  $Y_{Q(z)}^R$ .  $\square$

In view of Proposition 3.10,  $Y_{Q(z)}^L$  and  $Y_{Q(z)}^R$  give rise to a well defined linear map

$$Y_{Q(z)} = Y_{Q(z)}^L \otimes Y_{Q(z)}^R : V \otimes V \rightarrow (\text{End } \mathcal{D}_{Q(z)}(W))[[x, x^{-1}]]. \quad (3.49)$$

In particular,

$$Y_{Q(z)}(u \otimes v, x) = Y_{Q(z)}^L(u, x) Y_{Q(z)}^R(v, x) \quad (3.50)$$

for  $u, v \in V$ . Combining Theorem 3.5 with Propositions 3.11 and 3.10, using the arguments of Proposition 4.6.1 of [FHL] we immediately have:

**Theorem 3.12** *Let  $W$  be a (left) weak  $V$ -module and let  $z$  be a nonzero complex number. Then the pair  $(\mathcal{D}_{Q(z)}(W), Y_{Q(z)})$  carries the structure of a weak  $V \otimes V$ -module.  $\square$*

It is routine to check that the notion of  $\mathcal{D}_{Q(z)}(W)$  in the obvious way gives rise to a (contravariant) functor from the category of weak  $V$ -modules to the category of weak  $V \otimes V$ -modules.

## 4 $V \otimes V$ -homomorphisms and $Q(z)$ -intertwining maps

In this section, we shall prove that for  $V$ -modules  $W, W_1$  and  $W_2$ , a  $Q(z)$ -intertwining map of type  $\binom{W'}{W_1 W_2}$  in the sense of [HL2] exactly amounts to a  $V \otimes V$ -homomorphism from  $W_1 \otimes W_2$  to  $\mathcal{D}_{Q(z)}(W)$  and that a  $Q(z)$ -tensor product of  $V$ -modules  $W_1$  and  $W_2$  in the sense of [HL2] exactly amounts to a universal from  $W_1 \otimes W_2$  to a functor  $\mathcal{F}_{Q(z)}$ , which is essentially the composition of the functor  $\mathcal{D}_{Q(z)}$  with the contragredient module functor.

Following [HL2], for any  $\mathbb{C}$ -graded vector space  $U = \coprod_{h \in \mathbb{C}} U_{(h)}$ , we define the restricted dual

$$U' = \prod_{h \in \mathbb{C}} U_{(h)}^* \quad (4.1)$$

and the formal completion

$$\overline{U} = \prod_{h \in \mathbb{C}} U_{(h)}. \quad (4.2)$$

Then

$$\overline{U'} = U^*. \quad (4.3)$$

Furthermore, if  $\dim U_{(h)} < \infty$  for  $h \in \mathbb{C}$ , e.g.,  $U = W$  is a  $V$ -module, we have

$$\overline{U} = (U')^*. \quad (4.4)$$

Let  $W_1, W_2$  and  $W_3$  be  $V$ -modules. A  $Q(z)$ -intertwining map of type  $\binom{W_3}{W_1 W_2}$  (see [HL2]) is a linear map  $F : W_1 \otimes W_2 \rightarrow \overline{W_3}$  such that

$$\begin{aligned} & z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) Y^o(v, x_0) F(w_{(1)} \otimes w_{(2)}) \\ &= x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) F(Y^o(v, x_1) w_{(1)} \otimes w_{(2)}) \\ & \quad - x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) F(w_{(1)} \otimes Y(v, x_1) w_{(2)}) \end{aligned} \quad (4.5)$$

for  $v \in V$ ,  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ . (With  $W_3$  being a  $V$ -module, the action of  $Y^o(v, x)$  on  $W_3$  extends to  $\overline{W_3}$ .)

Following [HL2], we choose  $\log z$  so that

$$\log z = \log |z| + i \arg z \quad \text{with } 0 \leq \arg z < 2\pi. \quad (4.6)$$

Arbitrary values of the log function will be denoted by

$$l_p(z) = \log z + 2p\pi i \quad (4.7)$$

for  $p \in \mathbb{Z}$ .

We next recall from [HL2] a connection between intertwining operators of type  $\binom{W'_1}{W_3 W_2}$  and  $Q(z)$ -intertwining maps of type  $\binom{W'_3}{W_1 W_2}$ . Fix an integer  $p$ . Let  $\mathcal{Y}$  be any intertwining operator of type  $\binom{W'_1}{W_3 W_2}$ . Define a linear map  $F_{\mathcal{Y}, p}$  from  $W_1 \otimes W_2$  to  $\overline{W_3} = W_3^*$  by

$$\langle F_{\mathcal{Y}, p}(w_{(1)} \otimes w_{(2)}), w_{(3)} \rangle_{W_3} = \langle \mathcal{Y}(w_{(3)}, e^{l_p(z)} w_{(2)}), w_{(1)} \rangle_{W_1} \quad (4.8)$$

for  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ ,  $w_{(3)} \in W_3$ . It was proved in [HL2] that  $F_{\mathcal{Y}, p}$  is a  $Q(z)$ -intertwining map of type  $\binom{W'_3}{W_1 W_2}$ . Furthermore we have ([HL2], Proposition 4.7):

**Proposition 4.1** *Let  $W_1, W_2$  and  $W_3$  be (ordinary)  $V$ -modules. For  $p \in \mathbb{Z}$ , the correspondence  $\mathcal{F}_p : \mathcal{Y} \mapsto F_{\mathcal{Y}, p}$  is a linear isomorphism from the space  $\mathcal{V}_{W_3 W_2}^{W'_1}$  of intertwining operators of the indicated type to the space  $\mathcal{M}[Q(z)]_{W_1 W_2}^{W'_3}$  of  $Q(z)$ -intertwining maps of the indicated type.*

Note that a  $Q(z)$ -intertwining map is a linear map from a  $V \otimes V$ -module  $W_1 \otimes W_2$  to  $\overline{W}$ . Let  $A$  be a  $V \otimes V$ -module and  $W$  a  $V$ -module. Then we may define a notion of  $Q(z)$ -intertwining map from  $A$  to  $\overline{W}$ . Denote by  $Y_1$  and  $Y_2$  the two commuting (weak)  $V$ -module structures on  $A$  obtained by identifying  $V$  with  $V \otimes \mathbb{C}$  and  $\mathbb{C} \otimes V$ , respectively. A  $Q(z)$ -intertwining map from  $A$  to  $\overline{W}$  to be a linear map  $F$  from  $A$  to  $\overline{W}$  such that for  $v \in V$ ,  $a \in A$ :

$$\begin{aligned} & z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) Y^o(v, x_0) F(a) \\ &= z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) F(a) Y(v, x_0) \\ &= x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) F(Y_1^o(v, x_1) a) - x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) F(Y_2(v, x_1) a). \end{aligned} \quad (4.9)$$

The following is our key theorem of this section.



**Theorem 4.2** *Let  $A$  be a  $V \otimes V$ -module,  $W$  a  $V$ -module and  $F$  a linear map from  $A$  into  $\overline{W} (= (W')^*)$ . Then  $F$  is a  $Q(z)$ -intertwining map from  $A$  to  $\overline{W}$  if and only if  $F$  is a  $V \otimes V$ -homomorphism from  $A$  into  $\mathcal{D}_{Q(z)}(W') \subset (W')^*$ .*

**Proof.** Assume that  $F$  is a  $V \otimes V$ -homomorphism from  $A$  into  $\mathcal{D}_{Q(z)}(W')$ . For  $v \in V$ ,  $a \in A$ , since  $F(a) \in \mathcal{D}_{Q(z)}(W')$ , using (3.22) with  $\alpha = F(a)$  and the fact that  $F$  is a  $V \otimes V$ -homomorphism we obtain

$$\begin{aligned} & z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) Y^o(v, x_0) F(a) \\ &= x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) (Y_{Q(z)}^L)^o(v, x_1) F(a) - x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) Y_{Q(z)}^R(v, x_1) F(a) \\ &= x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) F(Y_1^o(v, x_1)a) - x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) F(Y_2(v, x_1)a). \end{aligned} \quad (4.10)$$

This proves that  $F$  is a  $Q(z)$ -intertwining map.

Conversely, assume that  $F$  is a  $Q(z)$ -intertwining map. Let  $v \in V$ ,  $a \in A$ , and let  $l \in \mathbb{N}$  be such that  $x^l Y_2(v, x)a \in A[[x]]$ . Taking  $\text{Res}_{x_1} x_1^l$  from (4.9) and using the fundamental delta-function substitution properties we get

$$(x_0 + z)^l Y^o(v, x_0) F(a) = (x_0 + z)^l F(Y_1^o(v, x_0 + z)a). \quad (4.11)$$

Since  $Y_1^o(v, x_0 + z)a \in A((x_0^{-1}))$ , we have

$$(x_0 + z)^l F(Y_1^o(v, x_0 + z)a) \in \overline{W}((x_0^{-1})), \quad (4.12)$$

hence (from (4.11))

$$(x_0 + z)^l Y^o(v, x_0) F(a) \in \overline{W}((x_0^{-1})). \quad (4.13)$$

In view of Lemma 3.2 we have  $F(a) \in \mathcal{D}_{Q(z)}(W')$ .

Note that  $F(a)Y(v, x_0)w' = \langle Y^o(v, x_0)F(a), w' \rangle$  for  $w' \in W'$ . Combining (3.22) with  $\alpha = F(a)$  and (4.9) we obtain

$$\begin{aligned} & x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) (Y_{Q(z)}^L)^o(v, x_1) F(a) - x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) Y_{Q(z)}^R(v, x_1) F(a) \\ &= x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) F(Y_1^o(v, x_1)a) - x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) F(Y_2(v, x_1)a). \end{aligned} \quad (4.14)$$

Note that for  $w' \in W'$ ,

$$\begin{aligned} & \langle (Y_{Q(z)}^L)^o(v, x_1) F(a), w' \rangle, \quad \langle F(Y_1^o(v, x_1)a), w' \rangle \in \mathbb{C}((x_1^{-1})), \\ & \langle Y_{Q(z)}^R(v, x_1) F(a), w' \rangle, \quad \langle F(Y_2(v, x_1)a), w' \rangle \in \mathbb{C}((x)). \end{aligned}$$

Right after this theorem we shall prove a simple fact (Lemma 4.3) which together with (4.14) immediately implies

$$(Y_{Q(z)}^L)^o(v, x_0) F(a) = F(Y_1^o(v, x_0)a) \quad (4.15)$$

$$Y_{Q(z)}^R(v, x_1) F(a) = F(Y_2(v, x_1)a). \quad (4.16)$$

This implies that  $F$  is a  $V \otimes V$ -homomorphism.  $\square$

We now prove the following simple fact which was used above:

**Lemma 4.3** *Let  $z$  be a nonzero complex number and let  $f(x) \in \mathbb{C}((x^{-1}))$ ,  $g(x) \in \mathbb{C}((x))$  be such that*

$$x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) f(x_1) - x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) g(x_1) = 0. \quad (4.17)$$

*Then  $f(x) = g(x) = 0$ .*

**Proof.** By taking  $\text{Res}_{x_0}$  from (4.17) we get  $f(x_1) = g(x_1)$ . Consequently,

$$f(x) = g(x) \in \mathbb{C}((x^{-1})) \cap \mathbb{C}((x)) = \mathbb{C}[x, x^{-1}]. \quad (4.18)$$

Then from (4.17) using the fundamental delta-function identities we get

$$z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) f(x_1) = 0. \quad (4.19)$$

Taking  $\text{Res}_{x_1}$  and using the fundamental delta-function properties we get  $f(z + x_0) = 0$ . Thus  $f(x) = 0$  and  $g(x) = f(x) = 0$ .  $\square$

In view of Proposition 4.1 and Theorem 4.2 we immediately have:

**Corollary 4.4** *Let  $W, W_1$  and  $W_2$  be (ordinary)  $V$ -modules and let  $p \in \mathbb{Z}$ . Then  $\mathcal{F}_p$  gives rise to a linear isomorphism from  $\mathcal{V}_{W_1 W_2}^{W'}$  to  $\text{Hom}_{V \otimes V}(W_1 \otimes W_2, \mathcal{D}_{Q(z)}(W'))$ . In particular,*

$$N_{W_1 W_2}^{W'} = \dim \mathcal{V}_{W_1 W_2}^{W'} = \dim \text{Hom}_{V \otimes V}(W_1 \otimes W_2, \mathcal{D}_{Q(z)}(W')). \quad \square \quad (4.20)$$

Recall a notion from [HL2]. A *generalized  $V$ -module* is a weak  $V$ -module on which  $L(0)$  acts semisimply. Then for a generalized  $V$ -module  $W$  we have the  $L(0)$ -eigenspace decomposition:  $W = \coprod_{h \in \mathbb{C}} W_{(h)}$ . Thus, a generalized  $V$ -module satisfies all the axioms defining the notion of a  $V$ -module ([FLM], [FHL]) except the two grading restrictions on the homogeneous subspaces.

Let  $W$  be a  $V$ -module. We denote by  $R_{Q(z)}(W)$  the sum of (ordinary)  $V \otimes V$ -submodules of  $\mathcal{D}_{Q(z)}(W)$ , on which  $L(0) \otimes 1$  acts semisimply. Then  $R_{Q(z)}(W)$  is a generalized  $V \otimes V$ -module. It is clear that Theorem 4.2 and Corollary 4.4 still hold when  $\mathcal{D}_{Q(z)}(W')$  is replaced by  $R_{Q(z)}(W')$ .

Note that the functor  $\mathcal{F}^c$  with  $\mathcal{F}^c(W) = W'$  is a contravariant functor from the category of  $V$ -modules to itself and that  $\mathcal{D}_{Q(z)}(R_{Q(z)})$  is a contravariant functor from the category of  $V$ -modules to the category of weak (generalized)  $V \otimes V$ -modules. The composition of the two (contravariant) functors gives us a (covariant) functor from the category of  $V$ -modules to the category of weak (generalized)  $V \otimes V$ -modules.

**Definition 4.5** Define  $\mathcal{F}_{Q(z)}$  to be the (covariant) functor from the category of  $V$ -modules to the category of generalized  $V \otimes V$ -modules such that

$$\mathcal{F}_{Q(z)}(W) = R_{Q(z)}(W') \quad \text{for } W \in \text{ob } \mathcal{C}_V, \quad (4.21)$$

$$\mathcal{F}_{Q(z)}(f) = (f')^*|_{R_{Q(z)}(W'_1)} = \bar{f}|_{R_{Q(z)}(W'_1)} \quad \text{for } f \in \text{Hom}_V(W_1, W_2), W_1, W_2 \in \text{ob}(\mathcal{C}_V)$$

where  $\mathcal{C}_V$  by definition denotes the category of  $V$ -modules.

Next, we shall give a connection between the functor  $\mathcal{D}_{Q(z)}$  and the notion of the  $Q(z)$ -tensor product defined in [HL2-3]. Recall from [HL2] that a  $Q(z)$ -product of  $W_1$  and  $W_2$  is a  $V$ -module  $W_3$  together with a  $Q(z)$ -intertwining map  $F$  of type  $\begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}$  and it is denoted by  $(W_3, F)$ . Let  $(W_3, F)$  and  $(W_4, G)$  be two  $Q(z)$ -products of  $W_1$  and  $W_2$ . A *morphism* from  $(W_3, F)$  to  $(W_4, G)$  is a  $V$ -module map  $\eta$  from  $W_3$  to  $W_4$  such that

$$G = \bar{\eta} \circ F \quad (4.23)$$

where  $\bar{\eta}$  is the natural map from  $\overline{W_3}$  to  $\overline{W_4}$  uniquely extending  $\eta$ .

**Definition 4.6** [HL2,4] A  $Q(z)$ -tensor product of  $W_1$  and  $W_2$  is a  $Q(z)$ -product  $(W, F)$  such that for any  $Q(z)$ -product  $(W_3, G)$ , there is a unique morphism from  $(W, F)$  to  $(W_3, G)$ . The  $V$ -module  $W$  is called a  $Q(z)$ -tensor product module of  $W_1$  and  $W_2$ .

In view of Theorem 4.2, for a  $Q(z)$ -product  $(W, F)$  of  $V$ -modules  $W_1$  and  $W_2$ ,  $F$  is a morphism from  $W_1 \otimes W_2$  to  $R_{Q(z)}(W')$  in the category of generalized  $V \otimes V$ -modules. Then the condition for a  $Q(z)$ -product  $(W, F)$  of  $W_1$  and  $W_2$  being a  $Q(z)$ -tensor product amounts to that for any  $V$ -module  $W_3$  and any  $V \otimes V$ -homomorphism  $G$  from  $W_1 \otimes W_2$  to  $R_{Q(z)}(W'_3) = \mathcal{F}_{Q(z)}(W_3)$  there exists a unique  $V$ -homomorphism  $\eta$  from  $W$  to  $W_3$  such that

$$G = \bar{\eta} \circ F = (\eta')^* \circ F = \mathcal{F}_{Q(z)}(\eta) \circ F. \quad (4.24)$$

The later condition, in terms of a categorical notion (cf. [J]), exactly amounts to that  $(W, F)$  is a universal from  $W_1 \otimes W_2$  to the functor  $\mathcal{F}_{Q(z)}$ .

To summarize we have:

**Proposition 4.7** Let  $W_1, W_2$  be  $V$ -modules and let  $(W, F)$  be a  $Q(z)$ -product of  $W_1$  and  $W_2$ . Then  $(W, F)$  is a  $Q(z)$ -tensor product of  $W_1$  and  $W_2$  if and only if  $(W, F)$  is a universal from  $W_1 \otimes W_2$  to the functor  $\mathcal{F}_{Q(z)}$ .  $\square$

It was proved in [Li2] that if every  $V$ -module is completely reducible, then every  $V \otimes V$ -module on which  $L(0) \otimes 1$  acts semisimply is completely reducible, in particular,  $R_{Q(z)}(W)$  is completely reducible for every  $V$ -module  $W$ . Let  $S$  be a fixed complete set of representatives of equivalent classes of irreducible  $V$ -modules. Then it follows from [FHL] that  $\{W_1 \otimes W_2 \mid W_1, W_2 \in S\}$  is a complete set of representatives of equivalent classes of irreducible  $V \otimes V$ -modules. Combining this with Corollary 4.4, we immediately have:

**Proposition 4.8** Assume that  $V$  is rational in the sense of Huang-Lepowsky. Let  $W$  be a  $V$ -module and  $z$  a nonzero complex number. Then  $R_{Q(z)}(W)$  is a direct sum of finitely many irreducible  $V \otimes V$ -modules. In particular,  $R_{Q(z)}(W)$  is an (ordinary)  $V \otimes V$ -module.  $\square$

Let  $S$  be a fixed complete set of representatives of equivalent classes of irreducible  $V$ -modules. For a  $V$ -module  $W$ , we set

$$H_{Q(z)}(S, W) = \coprod_{W_1, W_2 \in S} \mathcal{M}[Q(z)]_{W_1 W_2}^W \otimes (W_1 \otimes W_2). \quad (4.25)$$

We consider  $H_{Q(z)}(S, W)$  as a generalized  $V \otimes V$ -module in the obvious way. In view of Theorem 4.2 we obtain a  $V \otimes V$ -homomorphism

$$\Psi_W[Q(z)] : H_{Q(z)}(S, W) \rightarrow R_{Q(z)}(W') \quad (4.26)$$

such that

$$\Psi_W[Q(z)](F \otimes w_{(1)} \otimes w_{(2)}) = F(w_{(1)} \otimes w_{(2)}) \quad (4.27)$$

for  $F \in \mathcal{M}[Q(z)]_{W_1 W_2}^W$ ,  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  with  $W_1, W_2 \in S$ . Then the same argument of Theorem 4.17 in [Li2] gives:

**Proposition 4.9** *Assume that every  $V$ -module is completely reducible. Then for any  $V$ -module  $W$ ,  $\Psi_W[Q(z)]$  is a  $V \otimes V$ -isomorphism onto  $R_{Q(z)}(W')$ . In particular, if  $V$  is rational in the sense of Huang-Lepowsky, then this is true and  $R_{Q(z)}(W')$  is a  $V \otimes V$ -module.  $\square$*

Set

$$H(S) = \coprod_{W \in S} W' \otimes W. \quad (4.28)$$

Since  $Y$  is an intertwining operator of type  $\left(\begin{smallmatrix} W \\ V W \end{smallmatrix}\right)$  for any  $V$ -module  $W$ , by Proposition 4.1 ([HL2], Proposition 4.7) the linear map  $F_W$  defined by

$$\langle F_W(w' \otimes w), v \rangle = \langle w', Y(v, z)w \rangle \quad (4.29)$$

for  $w' \in W'$ ,  $w \in W$ ,  $v \in V$ ,  $v \in V$  is a  $Q(z)$ -intertwining map of type  $\left(\begin{smallmatrix} V' \\ W' W \end{smallmatrix}\right)$ . It follows from Theorem 4.2 that  $F_W$  is a  $V \otimes V$ -homomorphism from  $W' \otimes W$  to  $R_{Q(z)}(V)$ . Then we have a  $V \otimes V$ -homomorphism  $\Psi[Q(z)]$  from  $H(S)$  to  $R_{Q(z)}(V)$  such that

$$\langle \Psi[Q(z)](w' \otimes w), v \rangle = \langle w', Y(v, z)w \rangle \quad (4.30)$$

for  $v \in V$ ,  $w' \in W'$ ,  $w \in W$  with  $W \in S$ . It is easy to show that  $\mathcal{V}_{VW}^W$  is one-dimensional for any irreducible  $V$ -module  $W$ . Then we immediately have:

**Proposition 4.10** *Assume that every  $V$ -module is completely reducible. Then  $R_{Q(z)}(V)$  is a  $V \otimes V$ -module and  $\Psi[Q(z)]$  is a  $V \otimes V$ -isomorphism onto  $R_{Q(z)}(V)$ . In particular, the assertion holds if  $V$  is rational in the sense of Huang-Lepowsky.  $\square$*

**Remark 4.11** We here give a detailed proof for  $\mathcal{V}_{VW}^W$  being one-dimensional for any irreducible  $V$ -module  $W$ . First, we prove that  $\mathcal{V}_{VW_1}^{W_2}$  is linearly isomorphic to  $\text{Hom}_V(W_1, W_2)$  for any  $V$ -modules  $W_1$  and  $W_2$ . Then it follows from Schur lemma (see [FHL]) that  $\mathcal{V}_{VW}^W$  is one-dimensional for any irreducible  $V$ -module  $W$ . Let  $\mathcal{Y}$  be an intertwining operator of type  $\begin{pmatrix} W_2 \\ VW_1 \end{pmatrix}$ . Then  $\mathcal{Y}(\mathbf{1}, x) \in \text{Hom}(W_1, W_2)$  because

$$\frac{d}{dx}\mathcal{Y}(\mathbf{1}, x) = \mathcal{Y}(L(-1)\mathbf{1}, x) = 0.$$

Since  $v_i\mathbf{1} = 0$  for  $v \in V$ ,  $i \geq 0$ , from the Jacobi identity we get

$$Y(v, x_1)\mathcal{Y}(\mathbf{1}, x) = \mathcal{Y}(\mathbf{1}, x)Y(v, x_1).$$

That is,  $\mathcal{Y}(\mathbf{1}, x) \in \text{Hom}_V(W_1, W_2)$ . Then we have a linear map  $E : \mathcal{Y} \mapsto \mathcal{Y}(\mathbf{1}, x)$  from  $\mathcal{V}_{VW_1}^{W_2}$  to  $\text{Hom}_V(W_1, W_2)$ . If  $\mathcal{Y}(\mathbf{1}, x) = 0$ , then it follows from the Jacobi identity of  $\mathcal{Y}$  and and creation property  $v_{-1}\mathbf{1} = v$  that  $\mathcal{Y}(v, x) = 0$  for all  $v \in V$ . Thus,  $E$  is injective. On the other hand, let  $f$  be a  $V$ -homomorphism from  $W_1$  to  $W_2$ . Then it is clear that  $f \circ Y$  is an intertwining operator of type  $\begin{pmatrix} W_2 \\ VW_1 \end{pmatrix}$  such that

$$E(f \circ Y) = (f \circ Y)(\mathbf{1}, x) = fY(\mathbf{1}, x) = f.$$

Then  $E$  is onto. Therefore,  $E$  is a linear isomorphism.

**Remark 4.12** Let  $W_1, W_2$  and  $W_3$  be (ordinary)  $V$ -modules. From the proof of Propositions 4.7 and 12.2 in [HL2,4],  $Q(z)$ -intertwining maps of type  $\begin{pmatrix} W_3 \\ W_1W_2 \end{pmatrix}$  and  $P(z)$ -intertwining maps of type  $\begin{pmatrix} W'_1 \\ W'_3W_2 \end{pmatrix}$  can be canonically related as follows: Let  $F$  be a linear map from  $W_1 \otimes W_2$  to  $\overline{W_3}$ . Define a linear map  $F'$  from  $W'_3 \otimes W_2$  to  $\overline{W'_1} = W_1^*$  by

$$\langle F'(w'_{(3)} \otimes w_{(2)}), w_{(1)} \rangle_{W_1} = \langle w'_{(3)}, F(w_{(1)} \otimes w_{(2)}) \rangle_{W_3} \quad (4.31)$$

for  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ ,  $w'_{(3)} \in W'_3$ . Then it is straightforward to check that  $F$  is a  $Q(z)$ -intertwining map of type  $\begin{pmatrix} W_3 \\ W_1W_2 \end{pmatrix}$  if and only if  $F'$  is a  $P(z)$ -intertwining map of type  $\begin{pmatrix} W'_1 \\ W'_3W_2 \end{pmatrix}$ .

## 5 Huang-Lepowsky's $Q(z)$ -tensor functor

In this section, we shall extend Huang-Lepowsky's  $Q(z)$ -tensor functor constructed in [HL2-3] for the category of  $V$ -modules to a functor  $T_{Q(z)}$  from the category of  $V \otimes V$ -modules to the category of  $V$ -modules. We then show that the functor  $\mathcal{F}_{Q(z)}$  constructed in Section 4 is a right adjoint of  $T_{Q(z)}$ . As mentioned in Introduction, in establishing the functor  $T_{Q(z)}$  one may follow [HL2-3] and presumably the main result (Theorem 5.7) follows from Huang-Lepowsky's arguments in [HL2-3] without any further efforts. We here choose to use a different approach by making use of the regular representations.

Let  $A$  be a weak  $V \otimes V$ -module for now and let  $A^*$  be the algebraic dual of  $A$ . We define a vertex operator map  $Y'_{Q(z)}$  from  $V$  to  $(\text{End } A^*)[[x, x^{-1}]]$  by (cf. [HL2], (5.4))

$$\begin{aligned} & \langle Y'_{Q(z)}(v, x_0)\lambda, a \rangle \\ = & \text{Res}_{x_1} x_0^{-1} \delta\left(\frac{x_1 - z}{x_0}\right) \langle \lambda, Y_1^o(v, x_1)a \rangle - \text{Res}_{x_1} x_0^{-1} \delta\left(\frac{z - x_1}{-x_0}\right) \langle \lambda, Y_2(v, x_1)a \rangle \end{aligned} \quad (5.1)$$

for  $v \in V$ ,  $\lambda \in A^*$ ,  $a \in A$ .

**Definition 5.1** An element  $\lambda$  of  $A^*$  is said to satisfy Huang-Lepowsky's  $Q(z)$ -compatibility condition (see [HL2-3]) if

(a) the lower truncation condition holds for all  $v \in V$ :

$$Y'_{Q(z)}(v, x)\lambda \in A^*((x)). \quad (5.2)$$

(b) the following Jacobi identity relation holds for all  $v \in V$ ,  $a \in A$ :

$$\begin{aligned} & z^{-1} \delta\left(\frac{x_1 - x_0}{z}\right) \langle Y'_{Q(z)}(v, x_0)\lambda, a \rangle \\ = & x_0^{-1} \delta\left(\frac{x_1 - z}{x_0}\right) \langle \lambda, Y_1^o(v, x_1)a \rangle - x_0^{-1} \delta\left(\frac{z - x_1}{-x_0}\right) \langle \lambda, Y_2(v, x_1)a \rangle. \end{aligned} \quad (5.3)$$

Clearly, all the linear functionals on  $A$  that satisfy the Huang-Lepowsky's  $Q(z)$ -compatibility condition form a subspace of  $A^*$ , which we denote by  $\tilde{T}'_{Q(z)}(A)$ .

**Remark 5.2** Note that if  $A = W_1 \otimes W_2$  for  $V$ -modules  $W_1, W_2$ , then  $Y'_{Q(z)}(v, x_0)$  is exactly the one defined in [HL2] and the Huang-Lepowsky's  $Q(z)$ -compatibility condition is exactly the one defined therein.

Let  $v \in V$ ,  $\lambda \in A^*$ ,  $a \in A$  and let  $l \in \mathbb{N}$  be such that  $x_1^l Y_2(v, x_1)a \in A[[x_1]]$ . Then from (5.1) we get (cf. [DL])

$$(x_0 + z)^l \langle Y'_{Q(z)}(v, x_0)\lambda, a \rangle = (x_0 + z)^l \langle \lambda, Y_1^o(v, x_0 + z)a \rangle. \quad (5.4)$$

Since the expression on the right-hand side lies in  $\mathbb{C}((x_0^{-1}))$ , so does the expression on the left-hand side. Furthermore, if  $\lambda \in \tilde{T}'_{Q(z)}(A)$ , using the lower truncation condition we obtain

$$(x_0 + z)^l \langle Y'_{Q(z)}(v, x_0)\lambda, a \rangle \in \mathbb{C}((x_0)) \cap \mathbb{C}((x_0^{-1})) = \mathbb{C}[x_0, x_0^{-1}]. \quad (5.5)$$

Consequently, the formal series  $\langle Y'_{Q(z)}(v, x_0)\lambda, a \rangle$  converges to a rational function in the subring  $\mathbb{C}[x_0, x_0^{-1}, (x_0 + z)^{-1}]$ .

The following is an equivalent form of the Huang-Lepowsky's  $Q(z)$ -compatibility condition:

**Lemma 5.3** *An element  $\lambda$  of  $A^*$  satisfies the Huang-Lepowsky's  $Q(z)$ -compatibility condition if and only if for every  $v \in V$ , there exists  $k \in \mathbb{N}$  such that*

$$(x_1 - z)^k \langle \lambda, Y_1^o(v, x_1)a \rangle = (x_1 - z)^k \langle \lambda, Y_2(v, x_1)a \rangle \quad (5.6)$$

for all  $a \in A$ .

**Proof.** Assume that  $\lambda$  satisfies the Huang-Lepowsky's  $Q(z)$ -compatibility condition. For  $v \in V$ , let  $k$  be a nonnegative integer such that  $x^k Y'_{Q(z)}(v, x)\lambda \in A^*[[x]]$ . By applying  $\text{Res}_{x_0} x_0^k$  to (5.3) we obtain (5.6).

Conversely, assume (5.6). It follows from (5.1) and (5.6) that

$$x_0^k Y'_{Q(z)}(v, x_0)\lambda \in A^*[[x_0]].$$

That is,  $\lambda$  satisfies the lower truncation condition. Furthermore, it is well known (cf. [DL, Li1]) that (5.6) and (5.1) imply (5.3). This proves that  $\lambda$  satisfies the Huang-Lepowsky's  $Q(z)$ -compatibility condition.  $\square$

Note that if (5.6) holds for some  $k$ , then (5.6) holds with  $k$  being replaced by any integer greater than  $k$ .

**Remark 5.4** With Lemma 5.3, it follows from [FLM], [FHL], [DL] that the formal series  $\langle \lambda, Y_1^o(v, x)a \rangle$  and  $\langle \lambda, Y_2(v, x)a \rangle$  absolutely converge in the domains  $|x| > |z|$  and  $0 < |x| < |z|$ , respectively, to a common rational function with only two possible poles at  $x = 0, z$  such that the order of the pole at  $x = z$  is universally bounded for all  $a \in A$ .

**Remark 5.5** Note that the order of the pole at  $x = 0$  for the rational function discussed in Remark 5.4 in general depends on  $a \in A$ . So, an element  $\lambda$  of  $\tilde{T}'_{Q(z)}(A)$  is not quite an element of  $\mathcal{D}_{P(z)}(A, Y_1)$ , nor an element of  $\mathcal{D}_{Q(-z)}(A, Y_2)$ . On the other hand, as illustrated in the proof of Theorem 5.7,  $\lambda$  can be considered as an element of  $\mathcal{D}_{P(z)}(U, Y_1)$  for any finitely generated  $V$ -submodule  $U$  of  $(A, Y_1)$ .

We have (cf. [HL2], Proposition 6.2):

**Proposition 5.6** *The subspace  $\tilde{T}'_{Q(z)}(A)$  of  $A^*$  is stable under the action of all components of  $Y'_{Q(z)}(v, x_0)$  for  $v \in V$ , i.e.,*

$$Y'_{Q(z)}(v, x_0)\lambda \in \tilde{T}'_{Q(z)}(A)[[x_0, x_0^{-1}]] \quad \text{for } \lambda \in \tilde{T}'_{Q(z)}(A). \quad (5.7)$$

**Proof.** Let  $\lambda \in \tilde{T}'_{Q(z)}(A)$  and let  $u, v \in V$ . Let  $k \in \mathbb{N}$  be such that all the following identities hold for all  $a \in A$ :

$$(x_1 - y)^k Y_1^o(u, x_1) Y_1^o(v, y)a = (x_1 - y)^k Y_1^o(v, y) Y_1^o(u, x_1)a, \quad (5.8)$$

$$(x_1 - y)^k Y_2(u, x_1) Y_2(v, y)a = (x_1 - y)^k Y_2(v, y) Y_2(u, x_1)a, \quad (5.9)$$

$$(y - z)^k \langle \lambda, Y_1^o(v, y)a \rangle = (y - z)^k \langle \lambda, Y_2(v, y)a \rangle, \quad (5.10)$$

which follow from weak commutativity and Lemma 5.3. Replacing  $a$  with  $Y_1^o(u, x_1)a$  and  $Y_2(u, x_1)a$  in (5.10) we get

$$(y - z)^k \langle \lambda, Y_1^o(v, y) Y_1^o(u, x_1) a \rangle = (y - z)^k \langle \lambda, Y_2(v, y) Y_1^o(u, x_1) a \rangle, \quad (5.11)$$

$$(y - z)^k \langle \lambda, Y_1^o(v, y) Y_2(u, x_1) a \rangle = (y - z)^k \langle \lambda, Y_2(v, y) Y_2(u, x_1) a \rangle. \quad (5.12)$$

Noting that the actions  $Y_1$  and  $Y_2$  of  $V$  commute, using the fundamental properties of delta function and these identities we get

$$\begin{aligned} & (x_0 + z - y)^k (y - z)^k \langle Y'_{Q(z)}(u, x_0) \lambda, Y_1^o(v, y) a \rangle \\ = & \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) (x_1 - y)^k (y - z)^k \langle \lambda, Y_1^o(u, x_1) Y_1^o(v, y) a \rangle \\ & - \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) (x_1 - y)^k (y - z)^k \langle \lambda, Y_2(u, x_1) Y_1^o(v, y) a \rangle \\ = & \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) (x_1 - y)^k (y - z)^k \langle \lambda, Y_1^o(v, y) Y_1^o(u, x_1) a \rangle \\ & - \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) (x_1 - y)^k (y - z)^k \langle \lambda, Y_1^o(v, y) Y_2(u, x_1) a \rangle \\ = & \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) (x_1 - y)^k (y - z)^k \langle \lambda, Y_2(v, y) Y_1^o(u, x_1) a \rangle \\ & - \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) (x_1 - y)^k (y - z)^k \langle \lambda, Y_2(v, y) Y_2(u, x_1) a \rangle \\ = & \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) (x_1 - y)^k (y - z)^k \langle \lambda, Y_1^o(u, x_1) Y_2(v, y) a \rangle \\ & - \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) (x_1 - y)^k (y - z)^k \langle \lambda, Y_2(u, x_1) Y_2(v, y) a \rangle \\ = & (x_0 + z - y)^k (y - z)^k \langle Y'_{Q(z)}(u, x_0) \lambda, Y_2(v, y) a \rangle. \end{aligned} \quad (5.13)$$

Let  $n \in \mathbb{Z}$  be arbitrarily fixed. Let  $r \in \mathbb{N}$  be such that

$$x_0^{n+r} Y'_{Q(z)}(u, x_0) \lambda \in A^*[[x_0]].$$

Then using (5.13) we get

$$\begin{aligned} & (y - z)^{r+2k} \text{Res}_{x_0} x_0^n \langle Y'_{Q(z)}(u, x_0) \lambda, Y_1^o(v, y) a \rangle \\ = & \sum_{i=0}^{r+k} \binom{r+k}{i} \text{Res}_{x_0} (y - z - x_0)^{r+k-i} x_0^{n+i} (y - z)^k \langle Y'_{Q(z)}(u, x_0) \lambda, Y_1^o(v, y) a \rangle \\ = & \sum_{i=0}^r \binom{r+k}{i} \text{Res}_{x_0} (y - z - x_0)^{r+k-i} x_0^{n+i} (y - z)^k \langle Y'_{Q(z)}(u, x_0) \lambda, Y_1^o(v, y) a \rangle \\ = & \sum_{i=0}^r \binom{r+k}{i} \text{Res}_{x_0} (y - z - x_0)^{r+k-i} x_0^{n+i} (y - z)^k \langle Y'_{Q(z)}(u, x_0) \lambda, Y_2(v, y) a \rangle \\ = & \text{Res}_{x_0} \sum_{i=0}^{r+k} \binom{r+k}{i} (y - z - x_0)^{r+k-i} x_0^{n+i} (y - z)^k \langle Y'_{Q(z)}(u, x_0) \lambda, Y_2(v, y) a \rangle \\ = & (y - z)^{r+2k} \text{Res}_{x_0} x_0^n \langle Y'_{Q(z)}(u, x_0) \lambda, Y_2(v, y) a \rangle. \end{aligned} \quad (5.14)$$



Then from Lemma 5.3 we get

$$\text{Res}_{x_0} x_0^n Y'_{Q(z)}(u, x_0) \lambda \in \tilde{T}'_{Q(z)}(A). \quad (5.15)$$

This completes the proof.  $\square$

We have (cf. [HL2], Theorem 6.1):

**Theorem 5.7** *The pair  $(\tilde{T}'_{Q(z)}(A), Y'_{Q(z)})$  carries the structure of a weak  $V$ -module.*

**Proof.** From the  $Q(z)$ -compatibility condition and Proposition 5.6 we have

$$Y'_{Q(z)}(v, x) \lambda \in \tilde{T}'_{Q(z)}(A)((x)) \quad \text{for } v \in V, \lambda \in \tilde{T}'_{Q(z)}(A).$$

From (5.1), clearly,  $Y'_{Q(z)}(\mathbf{1}, x) = 1$ . Then it remains to prove the Jacobi identity.

Let  $a \in A$ . Let  $U$  be the  $V$ -submodule of  $(A, Y_1)$  generated by  $a$ . Denote by  $p_U$  the natural projection from  $A^*$  to  $U^*$ . For  $v \in V$ , let  $l \in \mathbb{N}$  be such that  $x^l Y_2(v, x) a \in A[[x]]$ . Since  $U$  is generated from  $a$  by  $Y_1$  and since  $Y_1$  and  $Y_2$  commute, we have

$$x^l Y_2(v, x) b \in A[[x]] \quad \text{for all } b \in U. \quad (5.16)$$

For  $\lambda \in \tilde{T}'_{Q(z)}(A)$ , let  $k \in \mathbb{N}$  be such that (5.6) holds. Then

$$x_1^l (x_1 - z)^k \langle \lambda, Y_1^o(v, x_1) b \rangle = x_1^l (x_1 - z)^k \langle \lambda, Y_2(v, x_1) b \rangle \in \mathbb{C}[x_1] \quad (5.17)$$

for all  $b \in U$ . Thus  $p_U(\lambda) \in \mathcal{D}_{P(z)}(U, Y_1)$ . Recall (5.4) and (2.26):

$$(x_0 + z)^l \langle Y'_{Q(z)}(v, x_0) \lambda, b \rangle = (x_0 + z)^l \langle \lambda, Y_1^o(v, x_0 + z) b \rangle, \quad (5.18)$$

$$(x_0 + z)^l \langle Y_{P(z)}^L(v, x_0) p_U(\lambda), b \rangle = (x_0 + z)^l \langle \lambda, Y_1^o(v, x_0 + z) b \rangle. \quad (5.19)$$

Then

$$(x_0 + z)^l \langle Y'_{Q(z)}(v, x_0) \lambda, b \rangle = (x_0 + z)^l \langle Y_{P(z)}^L(v, x_0) p_U(\lambda), b \rangle, \quad (5.20)$$

which by multiplying both sides by  $(z + x_0)^{-l}$  gives

$$\langle Y'_{Q(z)}(v, x_0) \lambda, b \rangle = \langle Y_{P(z)}^L(v, x_0) p_U(\lambda), b \rangle. \quad (5.21)$$

That is,

$$p_U(Y'_{Q(z)}(v, x_0) \lambda) = Y_{P(z)}^L(v, x_0) p_U(\lambda) \quad (5.22)$$

for  $v \in V, \lambda \in \tilde{T}'_{Q(z)}(A)$ . Then for  $u, v \in V, b \in U$ ,

$$\begin{aligned} \langle Y'_{Q(z)}(u, x_1) Y'_{Q(z)}(v, x_2) \lambda, b \rangle &= \langle p_U(Y'_{Q(z)}(u, x_1) Y'_{Q(z)}(v, x_2) \lambda), b \rangle \\ &= \langle Y_{P(z)}^L(u, x_1) Y_{P(z)}^L(v, x_2) p_U(\lambda), b \rangle. \end{aligned} \quad (5.23)$$

(We are using Proposition 5.6.) Similarly,

$$\langle Y'_{Q(z)}(v, x_2)Y'_{Q(z)}(u, x_1)\lambda, b \rangle = \langle Y^L_{P(z)}(v, x_2)Y^L_{P(z)}(u, x_1)p_U(\lambda), b \rangle, \quad (5.24)$$

$$\langle Y'_{Q(z)}(Y(u, x_0)v, x_2)\lambda, b \rangle = \langle Y^L_{P(z)}(Y(u, x_0)v, x_2)p_U(\lambda), b \rangle. \quad (5.25)$$

Then the Jacobi identity for  $Y'_{Q(z)}$  applied to any element of  $U$ , in particular, to  $a$ , follows from the Jacobi identity for  $Y^L_{P(z)}$ .  $\square$

Denote by  $T'_{Q(z)}(A)$  the sum of all (ordinary)  $V$ -submodules of  $\tilde{T}'_{Q(z)}(A)$ . Then  $T'_{Q(z)}(A)$  is a generalized  $V$ -module.

**Remark 5.8** Let  $U_1, U_2$  be arbitrary vector spaces. Let  $f$  be a linear map from  $U_1$  to  $U_2^*$ . Then  $f^*$  is a linear map from  $(U_2^*)^*$  to  $U_1^*$ , hence  $f^*|_{U_2}$  is a linear map from  $U_2$  to  $U_1^*$ . A simple fact is that the map  $f \mapsto f^*|_{U_2}$  is a linear isomorphism from  $\text{Hom}(U_1, U_2^*)$  to  $\text{Hom}(U_2, U_1^*)$  with

$$(f^*|_{U_2})^*|_{U_1} = f \quad \text{for } f \in \text{Hom}(U_1, U_2^*). \quad (5.26)$$

For a  $V \otimes V$ -module  $A$  and a  $V$ -module  $W$ , denote by  $\mathcal{M}_{Q(z)}(A, W)$  the space of all  $Q(z)$ -intertwining maps from  $A$  to  $W$ . Then we have:

**Theorem 5.9** *Let  $A$  be a  $V \otimes V$ -module,  $W$  a  $V$ -module and  $f$  a  $Q(z)$ -intertwining map from  $A$  to  $\overline{W}$  ( $= (W')^*$ ). Then  $f^*$  restricted to  $W'$  ( $\subset ((W')^*)^*$ ) is a  $V$ -homomorphism from  $W'$  to  $T'_{Q(z)}(A)$  ( $\subset A^*$ ). Furthermore, the linear map  $f \mapsto f^*|_{W'}$  is a linear isomorphism from  $\mathcal{M}_{Q(z)}(A, W)$  onto  $\text{Hom}_V(W', T'_{Q(z)}(A))$ .*

**Proof.** Let  $f$  be a  $Q(z)$ -intertwining map from  $A$  to  $\overline{W}$ . In view of Remark 5.8,  $f^*|_{W'}$  is a linear map from  $W'$  to  $A^*$ .

Let  $v \in V$ ,  $w' \in W'$ ,  $a \in A$ . Let  $k \in \mathbb{N}$  be such that  $x^k Y(v, x_0)w' \in W'[[x]]$ . Note that  $k$  depends only on  $v$  and  $w'$ . By applying  $\text{Res}_{x_0} x_0^k$  to (4.9) we get

$$(x_1 - z)^k \langle f(Y_1^o(v, x_1)a), w' \rangle = (x_1 - z)^k \langle f(Y_2(v, x_1)a), w' \rangle. \quad (5.27)$$

Then

$$(x_1 - z)^k \langle f^*(w'), Y_1^o(v, x_1)a \rangle = (x_1 - z)^k \langle f^*(w'), Y_2(v, x_1)a \rangle. \quad (5.28)$$

By Lemma 5.3,  $f^*(w') \in \tilde{T}'_{Q(z)}(A)$ . Therefore  $f^*$  maps  $W'$  into  $\tilde{T}'_{Q(z)}(A)$ .

Furthermore, using (5.1) with  $\lambda = f^*(w')$  and (4.9) we obtain

$$\begin{aligned} & \langle Y'_{Q(z)}(v, x_0)f^*(w'), a \rangle \\ &= \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) \langle f^*(w'), Y_1^o(v, x_1)a \rangle - \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) \langle f^*(w'), Y_2(v, x_1)a \rangle \\ &= \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) \langle f(Y_1^o(v, x_1)a), w' \rangle - \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) \langle f(Y_2(v, x_1)a), w' \rangle \\ &= \langle Y^o(v, x_0)f(a), w' \rangle \\ &= \langle f(a), Y(v, x_0)w' \rangle \\ &= \langle f^*(Y(v, x_0)w'), a \rangle. \end{aligned} \quad (5.29)$$

Thus

$$Y'_{Q(z)}(v, x_0)f^*(w') = f^*(Y(v, x_0)w') \quad \text{for } v \in V, w' \in W'. \quad (5.30)$$

Therefore,  $f^*|_{W'}$  is a  $V$ -homomorphism. In view of Remark 5.8, the map  $f \mapsto f^*|_{W'}$  is an injective map from  $\text{Hom}_{Q(z)}(A, W)$  to  $\text{Hom}_V(W', T'_{Q(z)}(A))$ .

On the other hand, let  $g$  be a  $V$ -homomorphism from  $W'$  to  $\tilde{T}'_{Q(z)}(A) (\subset A^*)$ . Then we have a linear map  $g^*|_A$  from  $A$  to  $(W')^* = \overline{W}$ . Now we prove that  $g^*|_A$  is a  $Q(z)$ -intertwining map.

Let  $a \in A$ ,  $v \in V$ ,  $w' \in W'$ . Using (5.4) with  $\lambda = g(w')$  we get

$$\begin{aligned} & x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) \langle g^*(Y_1^o(v, x_1)a), w' \rangle - x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) \langle g^*(Y_2(v, x_1)a), w' \rangle \\ &= x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) \langle g(w'), Y_1^o(v, x_1)a \rangle - x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) \langle g(w'), Y_2(v, x_1)a \rangle \\ &= z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) \langle Y'_{Q(z)}(v, x_0)g(w'), a \rangle \\ &= z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) \langle g(Y(v, x_0)w'), a \rangle \\ &= z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) \langle g^*(a), Y(v, x_0)w' \rangle \\ &= z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) \langle Y^o(v, x_0)g^*(a), w' \rangle. \end{aligned} \quad (5.31)$$

This proves that  $g^*$  is a  $Q(z)$ -intertwining map from  $A$  to  $\overline{W}$ . Then the map  $f \mapsto f^*|_{W'}$  is a linear isomorphism.  $\square$

Combining Theorems 5.9 and 4.2 we immediately have:

**Corollary 5.10** *Let  $A$  be a  $V \otimes V$ -module and  $W$  a  $V$ -module. Let  $f$  be a  $V \otimes V$ -homomorphism from  $A$  to  $R_{Q(z)}(W')$  ( $\subset (W')^*$ ). Then  $f^*$  restricted to  $W'$  ( $\subset ((W')^*)^*$ ) is a  $V$ -homomorphism from  $W'$  to  $T'_{Q(z)}(A)$  ( $\subset A^*$ ). Furthermore, the linear map  $f \mapsto f^*|_{W'}$  is a linear isomorphism from  $\text{Hom}_{V \otimes V}(A, R_{Q(z)}(W'))$  onto  $\text{Hom}_V(W', T'_{Q(z)}(A))$ .  $\square$*

We now assume that  $V$  is rational in the sense of Huang and Lepowsky. Then for each  $V \otimes V$ -module  $A$ ,  $T'_{Q(z)}(A)$  is a direct sum of irreducible  $V$ -modules. Denote by  $\mathcal{C}_{V \otimes V}^o$  the category of  $V \otimes V$ -modules on which both  $L^1(0)$  and  $L^2(0)$  act semisimply. It follows immediately from ([Li2], Proposition 4.15) that any  $V \otimes V$ -module  $A$  from  $\mathcal{C}_{V \otimes V}^o$  is semisimple. Then each  $V \otimes V$ -module  $A$  from  $\mathcal{C}_{V \otimes V}^o$  is a direct sum of finitely many irreducible  $V \otimes V$ -modules. Because by assumption all the fusion rules are finite, it follows from Corollary 4.4 that for any  $V \otimes V$ -module  $A$  in  $\mathcal{C}_{V \otimes V}^o$  and  $V$ -module  $W$ ,  $\text{Hom}_{Q(z)}(A, W)$  is finite-dimensional. Then it follows from Theorem 5.9 that the multiplicity of  $W'$  in  $T'_{Q(z)}(A)$  is finite. Therefore,  $T'_{Q(z)}(A)$  is a  $V$ -module. To summarize, we have (cf. [HL2], Proposition 5.6):

**Proposition 5.11** *Assume that  $V$  is rational in the sense of Huang and Lepowsky. Let  $A$  be a  $V \otimes V$ -module in  $\mathcal{C}_{V \otimes V}^o$ . Then  $T'_{Q(z)}(A)$  is a  $V$ -module.  $\square$*

Assume that  $V$  is rational in the sense of Huang and Lepowsky. Then we have a contravariant functor  $T'_{Q(z)}$  from the category  $\mathcal{C}_{V \otimes V}^o$  of  $V \otimes V$ -modules to the category of  $V$ -modules. For a  $V \otimes V$ -module from  $\mathcal{C}_{V \otimes V}^o$ , we set

$$T_{Q(z)}(A) = (T'_{Q(z)}(A))'. \quad (5.32)$$

This gives us a coinvariant functor  $T_{Q(z)}$ .

**Remark 5.12** In the case that  $A = W_1 \otimes W_2$  where  $W_1, W_2$  are  $V$ -modules,  $T'_{Q(z)}(W_1 \otimes W_2)$  is exactly the  $V$ -module  $W_1 \boxtimes_{Q(z)} W_2$  constructed in [HL2-3], so that  $T_{Q(z)}(W_1 \otimes W_2)$  is exactly the Huang-Lepowsky's  $Q(z)$ -tensor product module of  $W_1$  with  $W_2$ .

Note that  $\text{Hom}_V(W_1, W_2)$  is canonically isomorphic to  $\text{Hom}_V(W'_2, W'_1)$  (see [HL2]). Using Corollary 5.10 we immediately have:

**Theorem 5.13** *Assume that  $V$  is rational in the sense of Huang-Lepowsky. Then the functor  $\mathcal{F}_{Q(z)}$  is a right adjoint of the functor  $T_{Q(z)}$ .  $\square$*

Let  $A$  be a  $V \otimes V$ -module and  $W$  a  $V$ -module. An intertwining operator from  $A$  to  $W$  is a linear map  $F_x$  from  $A$  to  $W((x))$  such that the following conditions hold for  $v \in V, a \in A$ :

$$\begin{aligned} & x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(v, x_1) F_{x_2}(a) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) F_{x_2}(Y_2(v, x_1)a) \\ &= x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) F_{x_2}(Y_1(v, x_0)a) \end{aligned} \quad (5.33)$$

and

$$F_x(L_1(-1)a) = \frac{d}{dx} F_x(a). \quad (5.34)$$

Denote by  $\mathcal{V}(A, W)$  the space of all intertwining operators from  $A$  to  $W$ .

**Remark 5.14** If  $A = W_1 \otimes W_2$  for  $V$ -modules  $W_1$  and  $W_2$ , then the same proof with necessary adjustments of [HL4] shows that  $\mathcal{M}_{Q(z)}(A, W)$  is canonically isomorphic to  $\mathcal{V}(A, W)$  for any  $V$ -module  $W$ . But, for a general  $V \otimes V$ -module  $A$  from the category  $\mathcal{C}_{V \otimes V}^o$ , it is not easy to construct a canonical linear isomorphism between  $\mathcal{V}(A, W)$  and  $\mathcal{M}_{Q(z)}(A, W)$ .

## 6 Huang-Lepowsky's $P(z)$ -tensor functor

This section is parallel to Section 5. We here shall extend Huang-Lepowsky's  $P(z)$ -tensor functor constructed in [HL4] for the category of  $V$ -modules to a functor  $T_{P(z)}$  from the category  $\mathcal{C}_{V \otimes V}$  of  $V \otimes V$ -modules to the category of  $V$ -modules. In the construction of  $T_{P(z)}$  we exploit a relation between  $T_{P(z)}$  and  $T_{Q(-z^{-1})}$ . The idea of this approach is due to

[HL4], where the  $V$ -module structure of the  $P(z)$ -tensor product module was established by using the  $V$ -module structure of the  $Q(z^{-1})$ -tensor product module.

Let  $A$  be a weak  $V \otimes V$ -module for now,  $A^*$  the dual space. For  $v \in V$ ,  $\lambda \in A^*$  we define

$$Y'_{P(z)}(v, x)\lambda \in A^*[[x, x^{-1}]]$$

by

$$\begin{aligned} & \langle Y'_{P(z)}(v, x)\lambda, a \rangle \\ = & \text{Res}_{x_0} z^{-1} \delta \left( \frac{x^{-1} - x_0}{z} \right) \langle \lambda, Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)a \rangle \\ & + \text{Res}_{x_0} x_0^{-1} \delta \left( \frac{z - x^{-1}}{-x_0} \right) \langle \lambda, Y_2^o(v, x)a \rangle \end{aligned} \quad (6.1)$$

$$= \text{Res}_{x_0} z^{-1} \delta \left( \frac{x^{-1} - x_0}{z} \right) \langle \lambda, Y_1(e^{xL(1)}(-x^{-2})^{L(0)}v, x_0)a \rangle + \langle \lambda, Y_2^o(v, x)a \rangle \quad (6.2)$$

(cf. [HL4], (13.6)) for  $a \in A$ .

**Definition 6.1** A linear functional  $\lambda$  on  $A$  is said to satisfy Huang-Lepowsky's  $P(z)$ -compatibility condition if the following conditions hold:

- (a) The  $P(z)$ -lower truncation condition: For all  $v \in V$ , the formal Laurent series  $Y'_{P(z)}(v, x)\lambda$  involves only finitely many negative (integral) powers of  $x$ .
- (b) For all  $v \in V$ ,  $a \in A$ ,

$$\begin{aligned} & x_0^{-1} \delta \left( \frac{x_1^{-1} - z}{x_0} \right) \langle Y'_{P(z)}(v, x_1)\lambda, a \rangle \\ = & z^{-1} \delta \left( \frac{x_1^{-1} - x_0}{z} \right) \langle \lambda, Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}v, x_0)a \rangle \\ & + x_0^{-1} \delta \left( \frac{z - x_1^{-1}}{-x_0} \right) \langle \lambda, Y_2^o(v, x_1)a \rangle. \end{aligned} \quad (6.3)$$

Clearly, all the linear functionals on  $A$  that satisfy the Huang-Lepowsky's  $P(z)$ -compatibility condition form a subspace of  $A^*$ , which we denote by  $\tilde{T}'_{P(z)}(A)$ .

**Remark 6.2** In (6.1), by substituting  $x$  and  $v$  with  $x^{-1}$  and  $e^{xL(1)}(-x^{-2})^{L(0)}v$ , respectively, we obtain

$$\begin{aligned} & \langle (Y'_{P(z)})^o(v, x)\lambda, a \rangle \\ = & \text{Res}_{x_0} z^{-1} \delta \left( \frac{x - x_0}{z} \right) \langle \lambda, Y_1(v, x_0)a \rangle + \text{Res}_{x_0} x_0^{-1} \delta \left( \frac{z - x}{-x_0} \right) \langle \lambda, Y_2(v, x)a \rangle, \end{aligned} \quad (6.4)$$

noting that  $e^{-xL(1)}(-x^2)^{L(0)}e^{xL(1)}(-x^{-2})^{L(0)} = 1$  ([FHL], (5.3.1)). Similarly, in (6.3), by substituting  $x_1$  and  $v$  with  $x_1^{-1}$  and  $e^{x_1L(1)}(-x_1^{-2})^{L(0)}v$ , respectively, we obtain

$$\begin{aligned} & x_0^{-1}\delta\left(\frac{x_1-z}{x_0}\right)\langle(Y'_{P(z)})^o(v, x_1)\lambda, a\rangle \\ = & z^{-1}\delta\left(\frac{x_1-x_0}{z}\right)\langle\lambda, Y_1(v, x_0)a\rangle + x_0^{-1}\delta\left(\frac{z-x_1}{-x_0}\right)\langle\lambda, Y_2(v, x_1)a\rangle. \end{aligned} \quad (6.5)$$

Clearly, this is also equivalent to (6.3).

In the following, we shall construct a weak  $V \otimes V$ -module  $A_{new}$  from  $A$  such that

$$(\tilde{T}'_{P(z)}(A), Y'_{P(z)}) = (\tilde{T}'_{Q(-z^{-1})}(A_{new}), Y'_{Q(-z^{-1})}). \quad (6.6)$$

Then it will follow immediately from Theorem 5.7 that the pair  $(\tilde{T}'_{P(z)}(A), Y'_{P(z)})$  carries the structure of a weak  $V$ -module. The idea of this approach is due to [HL4], where the  $V$ -module structure of the  $P(z)$ -tensor product module was established by using the  $V$ -module structure of the  $Q(z^{-1})$ -tensor product module.

By replacing  $x_0$  with  $-zx_0x_1^{-1}$  in (6.3) we get

$$\begin{aligned} & -z^{-1}x_1x_0^{-1}\delta\left(\frac{-z^{-1}+x_1}{x_0}\right)\langle Y'_{P(z)}(v, x_1)\lambda, a\rangle \\ = & z^{-1}\delta\left(\frac{z^{-1}+x_0}{x_1}\right)\langle\lambda, Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}v, -zx_0/x_1)a\rangle \\ & -z^{-1}x_1x_0^{-1}\delta\left(\frac{x_1-z^{-1}}{x_0}\right)\langle\lambda, Y_2^o(v, x_1)a\rangle. \end{aligned} \quad (6.7)$$

That is,

$$\begin{aligned} & x_0^{-1}\delta\left(\frac{-z^{-1}+x_1}{x_0}\right)\langle Y'_{P(z)}(v, x_1)\lambda, a\rangle \\ = & x_0^{-1}\delta\left(\frac{x_1-z^{-1}}{x_0}\right)\langle\lambda, Y_2^o(v, x_1)a\rangle \\ & -x_1^{-1}\delta\left(\frac{z^{-1}+x_0}{x_1}\right)\langle\lambda, Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}v, -zx_0/x_1)a\rangle. \end{aligned} \quad (6.8)$$

Using the fundamental properties of delta function we get

$$\begin{aligned} & (-z^{-1})^{-1}\delta\left(\frac{x_0-x_1}{-z^{-1}}\right)\langle Y'_{P(z)}(v, x_1)\lambda, a\rangle \\ = & x_1^{-1}\delta\left(\frac{x_0+z^{-1}}{x_1}\right)\langle\lambda, Y_2^o(v, x_1)a\rangle \\ & -x_1^{-1}\delta\left(\frac{z^{-1}+x_0}{x_1}\right)\langle\lambda, Y_1(e^{x_1L(1)}(-x_1^{-2})^{L(0)}v, -zx_0/x_1)a\rangle. \end{aligned} \quad (6.9)$$

It is clear that (6.9) is equivalent to (6.3). Furthermore, (6.9) amounts to

$$\begin{aligned}
& (-z^{-1})^{-1} \delta \left( \frac{x_0 - x_1}{-z^{-1}} \right) \langle Y'_{P(z)}(v, x_1) \lambda, a \rangle \\
&= x_1^{-1} \delta \left( \frac{x_0 + z^{-1}}{x_1} \right) \langle \lambda, Y_2^o(v, x_0 + z^{-1}) a \rangle \\
&\quad - x_1^{-1} \delta \left( \frac{z^{-1} + x_0}{x_1} \right) \left\langle \lambda, Y_1 \left( e^{(z^{-1} + x_0)L(1)} (-(z^{-1} + x_0)^{-2})^{L(0)} v, \frac{-zx_0}{z^{-1} + x_0} \right) a \right\rangle
\end{aligned} \tag{6.10}$$

(cf. (5.3)).

The following results, formulated in [Li2] (Lemma 4.25, Remark 2.10), are special cases of the general result in change of variable ([Z1-2], [H2]):

**Lemma 6.3** *Let  $W$  be a weak  $V$ -module and  $z$  a nonzero complex number. For  $v \in V$ , set*

$$Y[z](v, x) = Y(z^{L(0)}v, zx). \tag{6.11}$$

*Then  $W$  equipped with  $Y[z]$  is a weak  $V$ -module. Furthermore, if  $W$  is a generalized  $V$ -module, then  $e^{L(0)\log z}$  is a  $V$ -isomorphism from  $(W, Y_W)$  to  $(W, Y[z])$ .*

**Lemma 6.4** *Let  $W$  be a weak  $V$ -module and  $z$  a nonzero complex number. For  $v \in V$ , set*

$$Y\{z\}(v, x) = Y \left( e^{-z(1+zx)L(1)} (1+zx)^{-2L(0)} v, \frac{x}{1+zx} \right). \tag{6.12}$$

*Then  $W$  equipped with  $Y\{z\}$  is a weak  $V$ -module and*

$$(Y\{z\})^o(v, x) = Y^o(v, x+z) \quad (= Y^o(v, y)|_{y=x+z}) \quad \text{for } v \in V. \tag{6.13}$$

*Furthermore, if  $W$  is a weak  $V$ -module on which  $L(1)$  acts semisimply, then  $e^{zL(1)}$  is a  $V$ -isomorphism from  $(W, Y_W)$  to  $(W, Y\{z\})$ .*

Furthermore, we also have:

**Lemma 6.5** *Let  $W$  be a weak  $V$ -module and  $z$  a nonzero complex number. For  $v \in V$ , set*

$$Y_{mix}(v, x) = Y \left( e^{(z^{-1}+x)L(1)} (z^{-1}+x)^{-2L(0)} (-1)^{L(0)} v, \frac{-zx}{z^{-1}+x} \right). \tag{6.14}$$

*Then  $W$  equipped with  $Y_{mix}$  is a weak  $V$ -module. Furthermore, if  $W$  is a generalized  $V$ -module on which  $L(1)$  acts locally nilpotently, then  $e^{(\pi i + \log z^2)L(0)} e^{zL(1)}$  is a  $V$ -isomorphism from  $(W, Y_W)$  to  $(W, Y_{mix})$ .*

**Proof.** Recall from [HL4] (cf. [FHL], (5.3.1)-(5.3.3)) that on  $V$ ,

$$z_0^{L(0)} e^{xL(1)} = e^{z_0^{-1}xL(1)} z_0^{L(0)} \quad (6.15)$$

for any nonzero complex number  $z_0$ . Then, for  $v \in V$ ,

$$\begin{aligned} Y_{mix}(v, x) &= Y \left( e^{(z^{-1}+x)L(1)} (z^{-1}+x)^{-2L(0)} (-1)^{L(0)} v, \frac{-zx}{z^{-1}+x} \right) \\ &= Y \left( (-z^2)^{L(0)} e^{-z(1+zx)L(1)} (1+zx)^{-2L(0)} v, \frac{-z^2x}{1+zx} \right) \\ &= Y[-z^2] \left( e^{-z(1+zx)L(1)} (1+zx)^{-2L(0)} v, \frac{x}{1+zx} \right) \\ &= (Y[-z^2])\{z\}(v, x), \end{aligned} \quad (6.16)$$

where we are using the notions in Lemmas 6.3 and 6.4. Then it follows from Lemmas 6.3 and 6.4 that  $(W, Y_{mix})$  is a weak  $V$ -module.

Assume that  $W$  is a generalized  $V$ -module on which  $L(1)$  acts locally nilpotently. Then

$$Y_{mix}(v, x) = e^{L(0) \log(-z^2)} e^{zL(1)} Y(v, x) e^{-zL(1)} e^{-L(0) \log(-z^2)}. \quad (6.17)$$

It is clear that the weak  $V$ -module structure  $(W, Y_{mix})$  is the transported weak  $V$ -module structure through the linear map  $e^{L(0) \log(-z^2)} e^{zL(1)}$ .  $\square$

Let  $\theta$  be the natural automorphism of  $V \otimes V$  uniquely determined by

$$\theta(u \otimes v) = v \otimes u \quad \text{for } u, v \in V. \quad (6.18)$$

Then we have a (weak)  $V \otimes V$ -module  $A^\theta$ , where  $A^\theta = A$  as a vector space and

$$Y_{A^\theta}(a, x) = Y_A(\theta(a), x) \quad \text{for } a \in V \otimes V. \quad (6.19)$$

**Proposition 6.6** *Let  $A$  be a weak  $V \otimes V$ -module and let  $z$  be a nonzero complex number. Let  $Y_{new}$  be the linear map from  $V \otimes V$  to  $(\text{End } A)[[x, x^{-1}]]$ , uniquely determined by*

$$\begin{aligned} Y_{new}(u \otimes v, x) &= Y_2 \left( e^{(z^{-1}+x)L(1)} (z^{-1}+x)^{-2L(0)} (-1)^{L(0)} u, \frac{-zx}{z^{-1}+x} \right) \\ &\quad \cdot Y_1 \left( e^{-z^{-1}(1+z^{-1}x)L(1)} (1+z^{-1}x)^{-2L(0)} (-1)^{L(0)} v, \frac{x}{1+z^{-1}x} \right) \end{aligned} \quad (6.20)$$

for  $u, v \in V$ . Then  $(A, Y_{new})$  carries the structure of a weak  $V \otimes V$ -module. Denote this weak  $V \otimes V$ -module by  $A_{new}$ . Furthermore, if  $A$  is a  $V \otimes V$ -module in the category  $\mathcal{C}_{V \otimes V}^o$ , then  $A_{new}$  is a  $V \otimes V$ -module in  $\mathcal{C}_{V \otimes V}^o$  and the linear map  $e^{L(0) \log(-z^2)} e^{zL(1)} \otimes e^{z^{-1}L(1)}$  is a  $V \otimes V$ -isomorphism from  $A^\theta$  to  $A_{new}$ .



**Proof.** For  $v \in V$ , set

$$Y_1^{new}(v, x) = Y_2 \left( e^{(z^{-1}+x)L(1)} (z^{-1} + x)^{-2L(0)} (-1)^{L(0)} v, \frac{-zx}{z^{-1} + x} \right), \quad (6.21)$$

$$Y_2^{new}(v, x) = Y_1 \left( e^{-z^{-1}(1+z^{-1}x)L(1)} (1 + z^{-1}x)^{-2L(0)} (-1)^{L(0)} v, \frac{x}{1 + z^{-1}x} \right). \quad (6.22)$$

By Lemmas 6.4 and 6.5,  $(A, Y_1^{new})$  and  $(A, Y_2^{new})$  are weak  $V$ -modules. Clearly,  $Y_1^{new}$  still commutes with  $Y_2^{new}$ . Then, the same proof of Proposition 4.6.1 of [FHL] shows that  $(A, Y_{new})$  is a weak  $V \otimes V$ -module. The rest immediately follows from Lemmas 6.4 and 6.5.  $\square$

In view of Lemma 6.4 and Proposition 6.6, combining (6.10) with (5.3) we immediately have:

**Proposition 6.7** *Let  $A$  be a weak  $V \otimes V$ -module and  $z$  a nonzero complex number. Then*

$$(\tilde{T}'_{P(z)}(A), Y'_{P(z)}) = (\tilde{T}'_{Q(-z^{-1})}(A_{new}), Y'_{Q(-z^{-1})}). \quad \square \quad (6.23)$$

Now, we have (cf. [HL4], Theorem 13.9):

**Theorem 6.8** *Let  $A$  be a weak  $V \otimes V$ -module. Then  $(\tilde{T}'_{P(z)}(A), Y'_{P(z)})$  carries the structure of a weak  $V$ -module. Furthermore, if  $A$  is a  $V \otimes V$ -module in the category  $\mathcal{C}_{V \otimes V}^o$ , then the linear map  $\psi^*$  suitably restricted is a  $V$ -isomorphism from  $(\tilde{T}'_{P(z)}(A), Y'_{P(z)})$  to  $(\tilde{T}'_{Q(-z^{-1})}(A^\theta), Y'_{Q(-z^{-1})})$ , where  $\psi$  is the linear automorphism of  $A$  defined by*

$$\psi(a) = (e^{L(0) \log(-z^2)} e^{zL(1)} \otimes e^{z^{-1}L(1)})a \quad \text{for } a \in A \quad (6.24)$$

(cf. [HL4], (13.19)) and  $\psi^*$  is the dual of  $\psi$ .

**Proof.** The first assertion immediately follows from Proposition 6.7 and Theorem 5.7. With Proposition 6.6, it follows from the functorial property that the restriction of  $\psi^*$  is a  $V$ -isomorphism from  $(\tilde{T}'_{Q(-z^{-1})}(A_{new}), Y'_{Q(-z^{-1})})$  to  $(\tilde{T}'_{Q(-z^{-1})}(A^\theta), Y'_{Q(-z^{-1})})$ . Then it follows from Proposition 6.7 that  $\psi^*$  is a  $V$ -isomorphism from  $(\tilde{T}'_{P(z)}(A), Y'_{P(z)})$  to  $(\tilde{T}'_{Q(-z^{-1})}(A^\theta), Y'_{Q(-z^{-1})})$ .  $\square$

Let  $A$  be a  $V \otimes V$ -module and  $W$  a  $V$ -module. A  $P(z)$ -intertwining map from  $A$  to  $W$  is a linear map  $F$  from  $A$  to  $\overline{W}$  such that for  $v \in V$ ,  $a \in A$ :

$$\begin{aligned} & x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) Y(v, x_1) F(a) - x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) F(Y_2(v, x_1)a) \\ &= z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) F(Y_1(v, x_0)a). \end{aligned} \quad (6.25)$$

Clearly, all  $P(z)$ -intertwining maps from  $A$  to  $W$  form a subspace of  $\text{Hom}(A, \overline{W})$ , which we denote by  $\mathcal{M}_{P(z)}(A, W)$ .

Let  $F_x$  be an intertwining operator map from  $A$  to  $W$ . Clearly, the evaluation of  $F_x$  at  $x = e^{\log z}$  is a  $P(z)$ -intertwining map from  $A$  to  $W$ . This gives rise to a linear map from  $\mathcal{V}(A, W)$  to  $\mathcal{M}_{P(z)}(A, W)$ .

The same proof of Proposition 12.2 in [HL4] (cf. [HL2], Proposition 4.7) gives:

**Proposition 6.9** *Let  $A$  be a  $V \otimes V$ -module from the category  $\mathcal{C}_{V \otimes V}^o$  and  $W$  a  $V$ -module. Then  $\mathcal{V}(A, W)$  is canonically isomorphic to  $\mathcal{M}_{P(z)}(A, W)$  through evaluation  $x = e^{\log z}$  in intertwining operators.  $\square$*

The same proof of Theorem 4.5 in [Li2] (cf. Theorem 4.2) gives:

**Proposition 6.10** *Let  $A$  be a  $V \otimes V$ -module and  $W$  a  $V$ -module. Then a  $P(z)$ -intertwining map from  $A$  to  $\overline{W}$  exactly amounts to a  $V \otimes V$ -homomorphism from  $A$  to  $R_{P(z)}(W') \subset \mathcal{D}_{P(z)}(W')$ .  $\square$*

The following theorem is parallel to Theorem 5.9:

**Theorem 6.11** *Let  $A$  be a weak  $V \otimes V$ -module,  $W$  a  $V$ -module and  $F$  a linear map from  $A$  to  $\overline{W}$ . Then  $F$  is a  $P(z)$ -intertwining map if and only if  $F^*|_{W'}$  is a  $V$ -homomorphism from  $W'$  to  $\tilde{T}'_{P(z)}(A) (\subset A^*)$ . Furthermore, the linear map  $F \mapsto F^*|_{W'}$  is a linear isomorphism from  $\mathcal{M}_{P(z)}(A, W)$  onto  $\text{Hom}_V(W', T'_{P(z)}(A))$ .*

**Proof.** Assume that  $F$  is a  $P(z)$ -intertwining map. Then for  $v \in V$ ,  $a \in A$ ,  $w' \in W'$ ,

$$\begin{aligned} & x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) \langle Y(v, x_1) F(a), w' \rangle - x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) \langle F(Y_2(v, x_1)a), w' \rangle \\ &= z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) \langle F(Y_1(v, x_0)a), w' \rangle. \end{aligned} \quad (6.26)$$

That is,

$$\begin{aligned} & x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) \langle F^*(Y^o(v, x_1)w'), a \rangle - x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) \langle F^*(w'), Y_2(v, x_1)a \rangle \\ &= z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) \langle F^*(w'), Y_1(v, x_0)a \rangle. \end{aligned} \quad (6.27)$$

By taking  $\text{Res}_{x_0}$  we get

$$\begin{aligned} & \langle F^*(Y^o(v, x_1)w'), a \rangle \\ &= \langle F^*(w'), Y_2(v, x_1)a \rangle + \text{Res}_{x_0} z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) \langle F^*(w'), Y_1(v, x_0)a \rangle. \end{aligned} \quad (6.28)$$

Combining this with (6.4) we obtain

$$\langle (Y'_{P(z)})^o(v, x_1) F^*(w'), a \rangle = \langle F^*(Y^o(v, x_1)w'), a \rangle, \quad (6.29)$$

which is equivalent to

$$Y'_{P(z)}(v, x_1) F^*(w') = F^*(Y(v, x_1)w'). \quad (6.30)$$

Since  $Y(v, x_1)w' \in W'((x))$ , it follows that  $F^*(w')$  satisfies the  $P(z)$ -lower truncation condition. Furthermore, (6.27) and (6.29) imply (6.5). In view of Remark 6.2,  $F^*(w') \in$

$\tilde{T}'_{P(z)}(A)$ . From (6.30),  $F^*|_{W'}$  is a  $V$ -homomorphism. In view of Remark 5.8,  $F \mapsto F^*|_{W'}$  is an injective map from  $\mathcal{M}_{P(z)}(A, W)$  to  $\text{Hom}_V(W', T'_{P(z)}(A))$ .

Let  $G$  be a  $V$ -homomorphism from  $W'$  into  $\tilde{T}'_{P(z)}(A)$  ( $\subset A^*$ ). For  $v \in V$ ,  $a \in A$ ,  $w' \in W$ , using (6.5) we get

$$\begin{aligned} & x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) \langle G(Y^o(v, x_1)w'), a \rangle \\ = & x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) \langle G(w'), Y_2(v, x_1)a \rangle + z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) \langle G(w'), Y_1(v, x_0)a \rangle. \end{aligned} \quad (6.31)$$

that is,

$$\begin{aligned} & x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) \langle G^*(a), Y^o(v, x_1)w' \rangle \\ = & x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) \langle G^*(Y_2(v, x_1)a), w' \rangle + z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) \langle G^*(Y_1(v, x_0)a), w' \rangle \end{aligned} \quad (6.32)$$

what is equivalent to,

$$\begin{aligned} & x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) \langle Y(v, x_1)G^*(a), w' \rangle \\ = & x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) \langle G^*(Y_2(v, x_1)a), w' \rangle + z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) \langle G^*(Y_1(v, x_0)a), w' \rangle \end{aligned} \quad (6.33)$$

That is,  $G^*|_A$  is a  $P(z)$ -intertwining map from  $A$  to  $\overline{W}$ . By Remark 5.8, we have  $(G^*|_A)^*|_{W'} = G$ . Therefore,  $F \mapsto F^*|_{W'}$  is a linear isomorphism from  $\mathcal{M}_{P(z)}(A, W)$  onto  $\text{Hom}_V(W', T'_{P(z)}(A))$ .  $\square$

Combining Proposition 6.10 with Theorem 6.11 we immediately have (cf. Corollary 5.10):

**Corollary 6.12** *Let  $A$  be a  $V \otimes V$ -module and  $W$  a  $V$ -module. Let  $f$  be a  $V \otimes V$ -homomorphism from  $A$  to  $R_{P(z)}(W')$  ( $\subset (W')^*$ ). Then  $f^*$  restricted to  $W'$  ( $\subset ((W')^*)^*$ ) is a  $V$ -homomorphism from  $W'$  to  $T'_{P(z)}(A)$  ( $\subset A^*$ ). Furthermore, the linear map  $f \mapsto f^*|_{W'}$  is a linear isomorphism from  $\text{Hom}_{V \otimes V}(A, R_{P(z)}(W'))$  on to  $\text{Hom}_V(W', T'_{P(z)}(A))$ .  $\square$*

With Theorem 6.11, using the same arguments of Proposition 5.11 we obtain:

**Proposition 6.13** *Assume that  $V$  is rational in the sense of Huang-Lepowsky. Then for any  $V \otimes V$ -module  $A$  in  $\mathcal{C}_{V \otimes V}^o$ ,  $T'_{P(z)}(A)$  is an (ordinary)  $V$ -module.  $\square$*

Assume that  $V$  is rational in the sense of Huang-Lepowsky. For a  $V \otimes V$ -module  $A$  in  $\mathcal{C}_{V \otimes V}^o$ , we set

$$T_{P(z)}(A) = (T'_{P(z)}(A))'. \quad (6.34)$$

This gives us a coinvariant functor  $T_{P(z)}$  from  $\mathcal{C}_{V \otimes V}^o$  to  $\mathcal{C}_V$ .

Immediately from Corollary 6.12 we have:

**Theorem 6.14** *Assume that  $V$  is rational in the sense of Huang-Lepowsky. Then the functor  $T_{P(z)}$  is a right adjoint of the functor  $\mathcal{F}_{P(z)}$ .  $\square$*

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